Centrum voor Wiskunde en Informatica

REPORT RAPPORT

MAS

Modelling, Analysis and Simulation

Modelling, Analysis and Simulation

Realization theory for rational systems

J. Němcová, J.H. van Schuppen

REPORT MAS-E0801 JANUARY 2008

Centrum voor Wiskunde en Informatica (CWI) is the national research institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organisation for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3703

Realization theory for rational systems

ABSTRACT
In this paper we solve the problem of realization of response maps for rational systems. Sufficient and necessary conditions for a response map to be realizable by a rational system are presented. The properties of rational realizations such as observability, controllability, and minimality are studied. Finally, we briefly discuss the procedures for checking observability and controllability of rational systems and minimality of rational realizations and the procedure for constructing a rational system realizing a response map.

2000 Mathematics Subject Classification: 93B15;93B25;93C10
Keywords and Phrases: rational systems;realization theory;minimality;controllability;observability
Note: This work was carried out under project MAS 2 - RATPOS.

Realization Theory for Rational Systems

Jana Němcová
Jan H. van Schuppen
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
Email J.Nemcova@cwi.nl, J.H.van.Schuppen@cwi.nl

December 21, 2007

In this paper we solve the problem of realization of response maps for rational systems. Sufficient and necessary conditions for a response map to be realizable by a rational system are presented. The properties of rational realizations such as observability, controllability, and minimality are studied. Finally, we briefly discuss the procedures for checking observability and controllability of rational systems and minimality of rational realizations and the procedure for constructing a rational system realizing a response map.

Mathematics Subject Classification (2000): 93B15, 93B25, 93C10.
Keywords and Phrases: rational systems, realization theory, minimality, controllability, observability.

1 Introduction

In this paper we present realization theory of rational systems.

The motivation to investigate realization theory of rational systems is the use of rational systems as models of phenomena in the life sciences, in particular in metabolic networks and in systems biology. For example, the well-known Michaelis-Menten system describing enzymatic reactions can be obtained by singular perturbation of a system with bilinear terms leading to a rational system which is even positive, i.e. a rational system such that the positive orthant is a forward invariant set. Rational systems also occur in optics and in economics. Moreover, as Bartosiewicz stated in [5], the theory of rational systems could be simpler and more powerful, once it is developed, than the theory of smooth systems.

The *realization problem* for rational systems considers a map from input functions to output functions and asks whether there exists a finite-dimensional rational system and an initial condition such that its input-output map is identical to the considered map. Such a system is then called a *realization* of the considered input-output map. A generalization, which is not addressed in this paper, is to regard any relation between observed variables and ask for a realization as a rational system.

Another goal of realization theory is to characterize certain properties of realizations. One wants to find the conditions under which the systems realizing the considered map are observable, controllable, or minimal. The relations between realizations having these properties are also of interest. Since controllability appears to be often an equivalent condition for the existence of a control law which achieves a particular control objective and since observability is an equivalent condition for the existence of an observer of a system, the realization theory is useful in control synthesis and observer synthesis. Because in system identification one wants to restrict attention to minimal realizations and because the parametrization of the class of systems, which is based on the parametrization of the set of minimal realizations using the equivalence relation of such realizations, is needed, studying minimal realizations and an equivalence relation between them within the realization theory finds its application as well.

The history of realization theory in system theory starts to be written by R.E. Kalman in [30] where he deals with realization of finite-dimensional linear systems. Of course, there is prior work on realization of automata. The generalization of realization theory from linear to nonlinear systems goes via bilinear systems to smooth and to analytic systems. For the realization of bilinear systems see for example [15]. There are three approaches to realization of nonlinear continuous-time systems described in [28]. See the references therein for Jakubczyk's approach, the approach by formal power series in non-commuting variables, and the Volterra series approach.

Polynomial and rational systems are a special case of nonlinear systems but within that class they admit a more refined algebraic structure. Realization theory of discrete-time polynomial systems was formulated by E.D. Sontag in [40]. Later, in [47], Y. Wang and E.D. Sontag published their results on realization theory for polynomial and rational continuous-time systems based on the approach of formal power series in non-commuting variables and on the relation of two characterizations of observation spaces. In [48] they generalize [47] to analytic realization theory and they relate it with the differential-geometric approach. In [49] the relation between orders of input/output equations and minimal dimensions of realizations is explored for both analytic and algebraic input/output equations. Another extension of [47] to the analytic case can be found in [45]. Further generalizations of [45] follow in [46].

Another approach to realization theory by polynomial continuous-time systems, motivated by the results of B. Jakubczyk in [29] for nonlinear realizations, is introduced by Z. Bartosiewicz in [3, 6]. He introduces, in [5], the concept of rational systems, but he does not solve the realization problem for this class of systems.

Our approach to the realization theory of rational systems is based on Z. Bartosiewicz's results presented in [5, 6]. Z. Bartosiewicz, in [5], introduces the concept of rational systems, which we adopt, and he studies the problem of immersion of smooth systems into rational systems. This problem is very similar to the problem of rational realization. Nevertheless, he just proposes the possibility to develop realization theory for rational systems, maybe with the help of his results on the immersion problem. Indeed, there is a strong analogy between our results and the results in [5] and [6].

Compared to the realization theory of rational systems developed by Y. Wang and E.D. Sontag in [47], our approach is different. We apply the so-called algebraic or algebraic geometric approach rather than the techniques, used in [47], based on formal power series. We solve the same problem of existence of rational realizations (compare Theorem 5.2 in [47] and Theorem 4.9 in this paper). Besides we deal with the questions of observability, controllability and minimality of rational realizations which are not treated in [47]. Another major difference is that the realizations within the class of rational systems which we consider do not have to be linear in the input as is assumed by Y. Wang and E.D. Sontag. This is motivated by the planned application of realization theory to biochemical systems where a glucose input may enter in a rational way.

The first step, motivated by biochemical reaction networks, in developing a realization theory for rational positive systems is done in [44]. We keep the problem of rational realization with the positivity constraint for further research.

It has been pointed out by M. Fliess and T. Glad, see [19, 20], that if the realization problems start from a relation on input-output functions with a specification of which external variables are inputs and which are outputs, then there may not exist a realization as a rational system. This situation is well known from linear systems where one works with descriptor systems or with behaviors in general. Therefore it would be appropriate to start the realization theory from behaviors, but in this paper that full generality of the realization problem is not treated. This generality is not needed for systems biology which motivates the study of rational systems.

The organization of the paper is as follows. Terminology, notation and mathematical preliminaries are provided in Section 2. Section 3 introduces the concept of rational systems which is, as we mentioned earlier, adopted from [5]. Trajectories, dimensions, controllability and observability of rational systems are also introduced. The problem of rational realization is formulated, and necessary and sufficient conditions for the existence of rational realization of a response map are presented in Section 4. We characterize the existence of a rational realization of a response map by the condition that a field determined by the considered map is finitely generated. The proof of a sufficient condition for a response map to be realizable by a rational system provides the procedure to construct a rational realization. Section 5 deals with canonical rational realizations. The equivalence relation between existence of rational, observable rational, and canonical, i.e. both observable and controllable, rational realizations is proved. As a consequence we get that each map realizable by a rational system can be realized by an observable rational realization, or even by a rational realization which is both observable and controllable. Minimal rational realizations are studied in Section 6. It is proved that canonical rational realizations are already minimal and that minimal rational realizations are controllable and observable if specific algebraic conditions are fulfilled. We also analyze the cases when a rational realization, which is not observable, is or is not minimal. Minimality and minimal-dimensionality are shown to be equivalent properties of rational realizations. The existence of a minimal rational realization for a response map is given by the existence of its rational realization. In Section 7 we study the relation of birational equivalence within the rational realizations of the same response map. Every rational realization which is birationally equivalent to a minimal rational realization of the same map is minimal. On the other hand, two canonical rational realization of the same response map are birationally equivalent. In Section 8 we propose, due to the theory developed in preceding sections, the procedures for checking observability of rational systems, minimality of rational realizations and for constructing a rational realization for a given map. The way how to develop a procedure for checking controllability of a rational system is shortly discussed as well. Section 9 concludes the paper.

2 Algebraic preliminaries

In this section we introduce the framework in which we formulate the problem of rational realization. The approach of algebraic geometry which relates algebraic properties of polynomial and rational functions and the objects of their geometric interpretation is very useful.

For the basic definitions and theorems of algebra and of algebraic geometry see [11, 12, 13, 26, 27, 34, 35, 50]. Although the definitions are the same, the notation may differ. Therefore we recall the basic notation below and otherwise, for more specific objects, in the rest of the paper.

By a polynomial in finitely many indeterminates $x_1, \ldots, x_n$ with real coefficients we mean a sum

$$\sum_{k \in \mathbb{N}^n} c(k) \prod_{i=1}^{n} x_i^{k(i)},$$

where just finitely many coefficients $c(k) \in \mathbb{R}$ are non-zero. We denote the ring of all polynomials in n variables with real coefficients by $\mathbb{R}[x_1, \ldots, x_n]$. The ring $\mathbb{R}[x_1, \ldots, x_n]$ can be also understood as an algebra over $\mathbb{R}$.

Because the field $\mathbb{R}$ is an integral domain, so is the algebra $\mathbb{R}[x_1, \ldots, x_n]$. Therefore we can define the field of quotients of $\mathbb{R}[x_1, \ldots, x_n]$ as the set of fractions $\{p/q | p, q \in \mathbb{R}[x_1, \ldots, x_n], q \neq 0\}$. This field of quotients is denoted by $\mathbb{R}(x_1, \ldots, x_n)$ and we refer to it also as to a field extension of $\mathbb{R}$. Generally we use the notation $\mathcal{Q}(S)$ for the field of quotients of an integral domain S. For example, $\mathcal{Q}(\mathbb{R}[x_1, \ldots, x_n]) = \mathbb{R}(x_1, \ldots, x_n)$.

Consider a ring R and an ideal $I \subseteq R$, we denote the quotient, respectively factor, ring of R modulo I by R/I.

Polynomial and rational functions on varieties

One of the basic objects in algebraic geometry is a variety. We will work with irreducible real affine varieties for two reasons. Firstly, irreducibility simplifies the technical details of the proofs. The case of general real affine varieties, not necessarily irreducible, is always reduced to a study of irreducible varieties and a simple study of several related irreducible varieties. Secondly, working with the real varieties allows us to have a better geometric understanding of the state spaces of rational systems.

A variety is *irreducible* if we cannot write it as an union of two non-empty varieties which are its strict subvarieties. A *real affine variety* is defined as an algebraic variety in an affine space $\mathbb{R}^n$, i.e. as a subset of $\mathbb{R}^n$ of zero points of finitely many polynomials with real coefficients in n variables.

We denote by X an arbitrary irreducible real affine variety in $\mathbb{R}^n$ with n finite, and by $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$ the ideal of all polynomials in n variables with real coefficients such that they are zero at every point from the variety X. By a *polynomial* on a variety X we mean a function $p : X \to \mathbb{R}$ for which there exists a polynomial $q \in \mathbb{R}[x_1, \ldots, x_n]$ such that $p = q$ on X. All polynomials from the set $q + I$ represent the same polynomial p on a variety X, but p (or the way how p is defined) is independent on the choice of its representant from $q + I$. Therefore p is well-defined. We denote by A the algebra of all polynomials on X.

So, distinct polynomials from A are in one-to-one correspondence with the equivalence classes of polynomials from $\mathbb{R}[x_1, \ldots, x_n]$ under congruence modulo I, i.e. A and the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/I$ are isomorphic. Due to the Hilbert Basis Theorem and the fact that the ideal in the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/I$ are in one-to-one correspondence with the ideals of $\mathbb{R}[x_1, \ldots, x_n]$ containing I, we know that every ideal in $\mathbb{R}[x_1, \ldots, x_n]/I$ is finitely generated. Therefore A is a finitely generated algebra of polynomials, in other words, there exist polynomial functions $\varphi_1, \ldots, \varphi_k \in A$ such that $A = \mathbb{R}[\varphi_1, \ldots, \varphi_k]$.

Since we assume that X is an irreducible variety, it follows that A is an integral domain. Therefore we can define Q, the field of quotients of A. Again, the elements of Q do not depend on the choice of the representants for polynomials in A. Moreover, the generators of A can be considered generators of Q. Therefore the field Q is finitely generated by the polynomials $\varphi_1, \ldots, \varphi_k \in A$, i.e. $Q = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$. We say that Q is finitely generated field extension of $\mathbb{R}$. The elements of Q are called *rational functions* on X. Even if $\varphi \in Q$ does not have to be defined on all of X, we write $\varphi : X \to \mathbb{R}$.

The $\mathbb{R}^n$ space can be endowed by the *Zariski topology* which is a topology where the closed sets are defined as real affine varieties. On the variety $X \subseteq \mathbb{R}^n$ the related topology is considered. We refer to it as to the Zariski topology on X. To emphasize the fact that we consider an open/closed/dense set in Zariski topology we call them Z-open/Z-closed/Z-dense. More details can be found in [26, 34].

Rational vector fields

Another thing which we need to know to define rational systems properly is how to differentiate rational functions on a variety.

Let X be an irreducible real affine variety, let A be the algebra of polynomials on X, and let Q denote the field of rational functions on X.

Definition 2.1 *A* rational vector field f *on* X *is a derivation of the field* Q*, i.e. an* $\mathbb{R}$*-linear map* $f : Q \longrightarrow Q$ *such that for* $\varphi, \psi \in Q$*,*

$$f(\varphi \cdot \psi) = f(\varphi) \cdot \psi + \varphi \cdot f(\psi).$$

The rational vector field f is well-defined at the point $x \in X$ if it maps the ring $O_x \subseteq Q$ of rational functions well-defined at x to itself. The set of all these points is denoted by $X(f)$, i.e. $X(f) = \{x \in X | f(O_x) \subseteq O_x\}$.

Example 2.2 *Let* $X = \mathbb{R}$ *and* $x_0 = 1$*. All rational functions on* X *defined at* x_0 *are the rational functions such that their denominator is not divisible by the term* $(x-1)$*. For example* $\frac{x^4-3x+1}{x+5} \in O_{x_0}$*, but* $\frac{x}{x^2-2x+1} \notin O_{x_0}$*. Let* $f = \frac{1}{x-1}\frac{\partial}{\partial x}$ *be the rational vector field on* X*. Then* $f(O_{x_0}) \subseteq Q \setminus O_{x_0}$ *and therefore* $x_0 \notin X(f)$*.*

Example 2.3 *Let* $X = \mathbb{R}^2$*,* $x_0 = (0,0)$ *and* $f = \frac{1}{(1+x+y)^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$*. Then* $O_{x_0} = \{\frac{p(x,y)}{q(x,y)}; p, q \in A, q(0,0) \neq 0\}$ *and* $f(O_{x_0}) \subseteq O_{x_0}$*.*

Definition 2.4 *The* trajectory of a rational vector field f *from a point* $x_0 \in X(f)$ *is the map* $x : [0,T) \to X(f) \subseteq X$ *such that for* $t \in [0,T)$ *and* $\varphi \in A$*,*

$$\frac{d}{dt}(\varphi \circ x)(t) = (f\varphi)(x(t)) \quad \textit{and} \quad \varphi(x(0)) = x_0.$$

The rational vector field f from the definition above works in the world of rational functions $\varphi \in O_{x_0}$. Generally, we cannot consider $\varphi \in Q$, because $f(\varphi) \notin O_{x_0}$ for $\varphi \in Q \setminus O_{x_0}$. Then the question arises: Why do we define the trajectory of a rational vector field f just by considering $\varphi \in A$? Obviously, it is simpler to consider $\varphi \in A$ instead of $\varphi \in O_{x_0}$ and moreover it is sufficient as follows from the proposition below.

Proposition 2.5 *Let* $x : [0,T) \to X$ *be a trajectory of a rational vector field* f *from a point* $x_0 \in X(f)$*. Consider* $t \in [0,T)$ *and* $\varphi \in O_{x_0}$*. Then*

$$\frac{d}{dt}(\varphi \circ x)(t) = (f\varphi)(x(t)).$$

Proof: Consider an arbitrary $\varphi \in O_{x_0} \subseteq Q$. Then $\varphi = \frac{\varphi_{num}}{\varphi_{den}}$ where $\varphi_{num}, \varphi_{den} \in A$ and $\varphi_{den} \neq 0$ at x_0. Note that for every $\theta \in A$ such that $\theta \neq 0$ at x_0 it holds that

$$f(\frac{1}{\theta}) = f(\theta\frac{1}{\theta^2}) = f(\theta)\frac{1}{\theta^2} + \theta f(\frac{1}{\theta^2}) = \frac{f(\theta)}{\theta^2} + \theta(2\frac{1}{\theta}f(\frac{1}{\theta})) = \frac{f(\theta)}{\theta^2} + 2f(\frac{1}{\theta}).$$

Therefore we get for φ_{den} the useful relation

$$f(\frac{1}{\varphi_{den}}) = -\frac{f(\varphi_{den})}{\varphi_{den}^2}. \tag{1}$$

Because f is a rational vector field, it follows by Definition 2.1 that

$$(f\varphi)(x(t)) = (f\frac{\varphi_{num}}{\varphi_{den}})(x(t)) = \left(f(\varphi_{num})\frac{1}{\varphi_{den}} + \varphi_{num}f(\frac{1}{\varphi_{den}})\right)(x(t)).$$

Then we get the equality $(f\varphi)(x(t)) = \left(\frac{f(\varphi_{num})\varphi_{den}-\varphi_{num}f(\varphi_{den})}{\varphi_{den}^2}\right)(x(t))$ by applying the relation (1). It is further rewritten as

$$\begin{aligned}(f\varphi)(x(t)) &= \frac{f(\varphi_{num})(x(t))\varphi_{den}(x(t)) - \varphi_{num}(x(t))f(\varphi_{den})(x(t))}{\varphi_{den}^2(x(t))} \\ &= \frac{\frac{d}{dt}(\varphi_{num}\circ x)(t)\varphi_{den}(x(t)) - \varphi_{num}(x(t))\frac{d}{dt}(\varphi_{den}\circ x)(t)}{\varphi_{den}^2(x(t))}.\end{aligned}$$

Consequently, by Definition 2.4, the desired equality appears to be derived as

$$(f\varphi)(x(t)) = \frac{d}{dt}\frac{\varphi_{num}\circ x}{\varphi_{den}\circ x}(t) = \frac{d}{dt}(\varphi\circ x)(t).$$

□

Theorem 2.6 *For any rational vector field f and any point $x_0 \in X(f)$ there exists an unique trajectory of f from x_0 defined on the maximal interval $[0,T)$ (T may be infinite).*

Proof: One way of proving this theorem is to follow the ideas of the Bartosiewicz's proof of the same statement for polynomial vector fields, see [4]. Another way is to use already proved more general statement about the existence of trajectories of smooth vector fields, see [1]. In both cases we transform the problem into $\mathbb{R}^n$. □

Transcendence degree

We define the algebraic independence of polynomial and rational functions on an irreducible real affine variety.

See for example [13] for a definition of the algebraic independence of polynomials on a variety which is not necessarily an irreducible real affine variety. We modify this definition to fit our framework.

Note that the algebraic independence of the rational functions on a variety is defined in the same way as the algebraic independence of polynomials. Because we will need to know how the algebraic independence is defined for field extensions of $\mathbb{R}$ and because the field of rational functions on an irreducible real affine variety is a field extension of $\mathbb{R}$ as well, we state the definition of algebraic independence for rational functions in more general setting. We define the transcendence degree of a field extension of $\mathbb{R}$ as the maximal number of algebraically independent elements of this extension. A transcendence basis of a field extension of $\mathbb{R}$ is then defined as a set of algebraically independent elements of that field whose cardinality equals the transcendence degree of the field. Again, we do not provide these definitions in the most general form. We make them more specific to suit our framework. For more details on transcendence degree, basis and extensions see [11, 12, 13, 34, 35, 50].

The notation and terminology introduced in the introduction of this section and in the subsection on polynomial and rational function on varieties is used onward.

Definition 2.7

(a) *Let X be an irreducible real affine variety and let A denote the algebra of polynomials on X. We call the elements $\varphi_1, \ldots, \varphi_s \in A$* algebraically independent *over $\mathbb{R}$ if there does not exist a non-zero polynomial p of s variables with real coefficients such that $p(\varphi_1, \ldots, \varphi_s) = 0$ in A.*

(b.1) *We call the elements $\varphi_1, \ldots, \varphi_s$ of a field extension of $\mathbb{R}$* algebraically independent *over $\mathbb{R}$ if there does not exist a non-zero polynomial p of s variables with real coefficients such that $p(\varphi_1, \ldots, \varphi_s) = 0$.*

(b.2) *Let F be a field extension of $\mathbb{R}$. We denote by* trdeg F *the* transcendence degree *of F over $\mathbb{R}$ which is defined as the largest number of elements of F which are algebraically independent over $\mathbb{R}$. An arbitrary subset of F of* trdeg F *algebraically independent elements is called a* transcendence basis *of F.*

Example 2.8 *Consider a variety $X = \mathbb{R}$. Then A denotes the set of all polynomials on the real axis. These are the polynomials of one variable with real coefficients on X.*

Let $\varphi_1 = x^4$ and $\varphi_2 = x - 1$ be two polynomials on X. Then the polynomial $p(\varphi_1, \varphi_2) = (\varphi_2 + 1)^4 - \varphi_1 = (x - 1 + 1)^4 - x^4$ equals 0, which means that the polynomials φ_1 and φ_2 are algebraically dependent.

Example 2.9 *Consider a variety X given as a solution of the polynomial equation $x^2 - y^2 - 1 = 0$. It is the unit circle in $\mathbb{R}^2$.*

Then the polynomials $p(x, y)$ and $p(x, y) + q(x, y)(x^2 - y^2 - 1)$, where $p(x, y)$ and $q(x, y)$ are arbitrary polynomial functions of two variables with real coefficients, coincide on X. Therefore, any two polynomials p, q on X are algebraically independent if there does not exist a non-zero polynomial n in two variables with real coefficients such that $n(p(x, y), q(x, y)) = m(x, y)(x^2 - y^2 - 1)$ with m being a polynomial in two variables with real coefficients.

For example, $p(x, y) = x^2$ and $q(x, y) = y^2$ are algebraically dependent because if we substitute x^2 for $\tilde{x}$ and y^2 for $\tilde{y}$ in a non-zero polynomial $n(\tilde{x}, \tilde{y}) = \tilde{x}^2 - \tilde{y}^2 - 2\tilde{y} + 1$ we get that $n(p(x, y), q(x, y)) = (x^2 + y^2 + 1)(x^2 - y^2 - 1)$ which is a zero polynomial on X.

On the other hand, the polynomials $p(x, y) = xy$ and $q(x, y) = xy^4$ are algebraically independent because every polynomial combination of xy and xy^4

is not divisible by $x^2 - y^2 - 1$*. This can be checked by the following syntax in Maple:*

```
with(PolynomialIdeals):
J:=PolynomialIdeal(xy,xy^4);
f:=expand(x^2-y^2-1);
IdealMembership(f,J);
```

The output is then "true" or "false" depending on whether the polynomial $x^2 - y^2 - 1$ *is or is not contained in the ideal generated by the polynomials* xy *and* xy^4*.*

Example 2.10 *Consider a variety* $X = \mathbb{R}^3$ *and a field* $\mathbb{R}(\varphi_1, \varphi_2, \varphi_3)$ *generated by the rational functions*

$$\varphi_1 = \frac{x}{y+z}\ ,\quad \varphi_2 = \frac{x}{y-z}\ ,\quad \varphi_3 = \frac{y^2 - z^2}{x^2 y}.$$

From the equalities $\varphi_1\varphi_2\varphi_3 = \frac{1}{y}$ *and* $(\varphi_1 + \varphi_2)\varphi_3 = \frac{2}{x}$ *it follows that* $x, y \in \mathbb{R}(\varphi_1, \varphi_2, \varphi_3)$*. Because* $(\varphi_1 \frac{1}{x})^{-1} - y = z$*, we know that even* $z \in \mathbb{R}(\varphi_1, \varphi_2, \varphi_3)$*. Therefore,* $\mathbb{R}(\varphi_1, \varphi_2, \varphi_3) = \mathbb{R}(x, y, z)$ *and consequently,* $\operatorname{trdeg} \mathbb{R}(\varphi_1, \varphi_2, \varphi_3) = \operatorname{trdeg} \mathbb{R}(x, y, z) = 3$*. Then the transcendence basis of* $\mathbb{R}(\varphi_1, \varphi_2, \varphi_3)$ *can be taken as the set* $\{x, y, z\}$*.*

Definition 2.11 *Let* F *be a subfield of a field* G*. An element* $g \in G$ *is said to be* algebraic *over* F *if there exist elements* $f_0, \dots, f_j \in F$*,* $j \geq 1$*, not all equal to zero, such that*

$$f_0 + f_1 g + \cdots + f_j g^j = 0.$$

Because several properties of transcendence degree used within the paper were not found in the literature in the form we use them, we state these properties with the proofs below. Proposition 2.12 can be derived as a consequence of Proposition 2 from [12, Chapter 6.2] (the same statement as in Proposition 2 can be found in [51, Vol.1]) but the proof stated here is different. An equivalent of Proposition 2.13 was not found in literature. Theorem 28 from [51, Vol.1, Chapter II, p. 100] says that the transcendence degree of an integral domain which is a homeomorphic image of another integral domain is lower than the transcendence degree of its preimage. Note that a field is an integral domain but not other way round. Therefore, Proposition 2.14, which states the same statement as Theorem 28 but just for field extensions of $\mathbb{R}$, is its direct corollary. In spite of this we also provide the proof of this proposition since it is different from the proof of Theorem 28.

Proposition 2.12 *Let* F *be a subfield of a field* G*, i.e.* $F \subseteq G$*. Then* $\operatorname{trdeg} F \leq \operatorname{trdeg} G$*.*

Proof: We know from [35, Chapter X, Theorem 1] that any two transcendence bases of the same field have the same cardinality. So we can choose an arbitrary transcendence basis to have the same number of elements determining the transcendence degree. Moreover, from the same theorem, a transcendence basis can be chosen from a set of generators.

Since F is a subfield of a field G, we can assume that the set of generators of F is a subset of the set of generators of G. Hence, if we choose a transcendence basis S_F of F from a set of generators of F, we can find a transcendence basis S_G of G such that $S_F \subseteq S_G$. Therefore, directly from the definition of transcendence degree (Definition 2.7(b.2)), we get that trdeg $F \leq$ trdeg G. □

Proposition 2.13 *Let F be a subfield of a field G such that* trdeg $F =$ trdeg G. *If the elements of $G \setminus F$ are not algebraic over F then $F = G$.*

Proof: We denote by S_F a transcendence basis of F and by $f_1, \ldots, f_{\text{trdeg } F}$ the elements of S_F. Hence, $\{f_1, \ldots, f_{\text{trdeg } F}\} = S_F \subset F$. Since $F \subseteq G$ and trdeg $F =$ trdeg G we can assume without loss of generality that $S_F = S_G$.

To prove the equality $F = G$ we will prove the relation $G \subseteq F$ by contradiction. Assume that there exists $g \in G$ such that $g \notin F$. Since S_G is a maximal algebraically independent set of G, the set $S_G \cup \{g\}$ is algebraically dependent. Therefore there exists a non-zero polynomial p with real coefficients such that $p(f_1, \ldots, f_{\text{trdeg } F}, g) = 0$. The polynomial p can be rewritten in the form

$$p_0(f_1, \ldots, f_{\text{trdeg } F}) + p_1(f_1, \ldots, f_{\text{trdeg } F})g + \cdots + p_j(f_1, \ldots, f_{\text{trdeg } F})g^j = 0,$$

where $p_0, p_1, \ldots, p_j$ are polynomials of trdeg $F - 1$ variables such that

$$p_0(f_1, \ldots, f_{\text{trdeg } F}), \ldots, p_j(f_1, \ldots, f_{\text{trdeg } F}) \in F.$$

Hence, by Definition 2.11, g is algebraic over F. This is the contradiction with the assumption that all elements of $G \setminus F$ are not algebraic over F. Therefore $G \setminus F = \emptyset$ and because $F \subseteq G$, it implies that $F = G$. □

Proposition 2.14 *Let F and G be field extensions of $\mathbb{R}$ such that there exists a field isomorphism $i : F \to G$, $G = i(F)$. Then* trdeg $F =$ trdeg G.

Proof: We denote by $S_F = \{f_1, \ldots, f_{\text{trdeg } F}\}$ a transcendence basis of F. Since $f_1, \ldots, f_{\text{trdeg } F}$ are algebraically independent over $\mathbb{R}$, we know that for all non-zero polynomials p with real coefficients in trdeg F variables,

$$p(f_1, \ldots, f_{\text{trdeg } F}) \neq 0.$$

Consider the set $i(S_F) = \{i(f_1), \ldots, i(f_{\text{trdeg } F})\}$. Since the isomorphism i preserves sums and products, $p(i(f_1), \ldots, i(f_{\text{trdeg } F})) = i(p(f_1, \ldots, f_{\text{trdeg } F}))$.

If the image $i(p(f_1, \ldots, f_{\text{trdeg } F}))$ of a non-zero polynomial p would be zero, it would be a contradiction with the fact that i is injective. Therefore

$$p(i(f_1), \ldots, i(f_{\text{trdeg } F})) \neq 0,$$

for every non-zero polynomial p in trdeg F variables with real coefficients. The set $i(S_F)$ is therefore a subset of a transcendence basis of G. Thus, trdeg $F \leq$ trdeg G.

The converse inequality trdeg $F \leq$ trdeg G can be proved in the same way. We consider the inverse i^{-1} of an isomorphism $i : F \to G$, i.e. i^{-1} is an isomorphism such that $i^{-1}(G) = F$. Let $S_G = \{g_1, \ldots, g_{\text{trdeg } G}\}$ be a transcendence basis of G. Then $p(g_1, \ldots, g_{\text{trdeg } G}) \neq 0$ for all polynomials p with real coefficients in trdeg G variables and

$$i^{-1}(p(g_1, \ldots, g_{\text{trdeg } G})) = p(i^{-1}(g_1), \ldots, i^{-1}(g_{\text{trdeg } G})) \neq 0.$$

Hence, the set $i^{-1}(S_G)$ is a subset of a transcendence basis of F and trdeg $F \geq$ trdeg G. $\square$

3 Rational systems

We consider rational systems as systems on irreducible real affine varieties with the dynamics defined by rational vector fields and with rational output functions. This concept was introduced in [5] and we will explain it in the next subsection. The dependency of trajectories on inputs is stressed in the subsection dedicated to the trajectories of rational systems. New notation for a state trajectory reflexing this dependency is introduced as well as some properties of a map related to a trajectory. The subsection on dimensions of state spaces deals with proper definitions of dimension of varieties which are taken as the state spaces for rational systems. In Section 6, using this definition of dimension of rational systems, we define minimal rational realizations. In the subsection on controllability and observability of rational systems we define these two properties and illustrate them on a simple example. Just as controllability and observability, the observation algebra and the observation field of a response map defined afterwards are already introduced in [5].

Rational systems

We define an *input space* U as an arbitrary set of input values. We can take for example $U \subseteq \mathbb{R}^m$. For the *space of input functions* we restrict attention to the set $\mathcal{U}_{pc} = \{u : [0, T) \to U | u \text{ piecewise constant}\}$. To consider just piecewise constant functions as the space of input functions is not too restrictive because an arbitrary input function can be approximated by piecewise constant inputs and because the considered response maps are analytic at the switching time points, see below.

Let u be an input from $\mathcal{U}_{pc}$. Then $u = (\alpha_1, t_1)(\alpha_2, t_2) \ldots (\alpha_n, t_n)$ means that for $t \in [\sum_{j=0}^{i} t_j, \sum_{j=0}^{i+1} t_j)$ the input $u(t) = \alpha_{i+1} \in U$ for $i = 0, 1, \ldots, n-1$, $t_0 = 0$. We denote $T_u = \sum_{j=1}^{n} t_j$. If $u = (\alpha_1, t_1) \ldots (\alpha_n, t_n)$ and $v = (\beta_1, s_1) \ldots (\beta_k, s_k)$ are both inputs, then $(u)(v)$ is an input function which we get by concatenating v to u, i.e. $(u)(v) = (\alpha_1, t_1) \ldots (\alpha_n, t_n)(\beta_1, s_1) \ldots (\beta_k, s_k)$.

Every input function $u \in \mathcal{U}_{pc}$ has a time domain $[0, T_u)$. To express that the input u was applied just till time $t \in [0, T_u)$ we will write a subindex t to u like u_t. This means that u_t corresponds to the input u restricted to the time domain $[0, t) \subseteq [0, T_u)$. The empty input e is such that $T_e = 0$.

The output space is $\mathbb{R}^r$.

Definition 3.1 *A* rational system Σ *is a triple* (X, f, h) *where*

(i) X *is an irreducible real affine variety,*

(ii) $f = \{f_\alpha | \alpha \in U\}$ *is a family of rational vector fields on* X*,*

(iii) $h : X \to \mathbb{R}^r$ *is an output map with rational components.*

The assumption of irreducibility of the variety which we take as a state space provides that the coordinate ring A (see the subsection on polynomial and rational functions on varieties of Section 2) of that variety is an integral domain. This property allows us to take the factor space according to this ring.

From the state space X we consider just states at which the output function h is defined and at which at least one of the rational vector fields $\{f_\alpha | \alpha \in U\}$ is defined. The set of these points is a Z-dense subset of X.

Consider a rational system $\Sigma = (X, f, h)$ with an initial condition $x_0 \in X$ (such system can be directly denoted as $\Sigma = (X, f, h, x_0)$) and a fixed input function $u \in U_{pc}$. If there exists a trajectory of the system Σ as the trajectory of the rational vector field f determined by u from x_0 (see Definition 2.4), we denote it as $x(\cdot, x_0, u) : [0, T_u) \to X$.

Since a trajectory of system Σ with an initial condition $x_0 \in X$ does not need to exist for every input $u \in \mathcal{U}_{pc}$, we define the set of admissible inputs $\mathcal{U}_{pc}(x_0) = \{u \in \mathcal{U}_{pc} | x(\cdot, x_0, u) \text{ exists}\}$. Now, for every input $u \in \mathcal{U}_{pc}(x_0)$ there exists a trajectory $x(\cdot, x_0, u)$ of Σ with an initial condition $x_0 \in X$. We conclude the convention that a trajectory of a rational system Σ with an initial state $x_0 \in X$ is for the empty input e equal to x_0, i.e. $x(0, x_0, e) = x_0$. Then the set of admissible inputs $\mathcal{U}_{pc}(x_0)$ may contain just the empty input e, see Example 3.3. The set of all $x_0 \in X$ such that $\mathcal{U}_{pc}(x_0) \setminus \{e\} \neq \emptyset$ is a Z-dense open subset of X and we denote it by X_Σ.

Example 3.2 *Consider a rational system* $\Sigma = (X, f, h, x_0)$ *such that the state space* $X = \mathbb{R}$*, the initial state* $x_0 = 1$ *and the rational vector field* $f = \frac{1}{x-2+u} \frac{\partial}{\partial x}$ *on* X*. Obviously,* $x_0 \in X(f)$ *if* u *is an zero input because* $f(O_x)$ *for* $f = \frac{1}{x-2} \frac{\partial}{\partial x}$ *is a subset of* O_x*. Consider an input* $u \in \mathcal{U}_{pc}$ *such that* $u = 1$ *on an interval* $[0, \epsilon]$ *for an* $\epsilon > 0$*, i.e.* $u_\epsilon = 1$*. Then* $f = \frac{1}{x-2+1} \frac{\partial}{\partial x} = \frac{1}{x-1} \frac{\partial}{\partial x}$ *describes the rational vector field* f *till the time* ϵ*. We know from Example 2.2 that in this* *case $x_0 \notin X(f)$. So, the rational vector field f is not well-defined at x_0 and therefore the trajectory of f from x_0 for the input u does not exist. Hence $u \notin \mathcal{U}_{pc}(x_0)$.*

Example 3.3 *Consider a rational system $\Sigma = (X, f, h, x_0)$. Let $X = \mathbb{R}$, $x_0 = 1$ and $f = \frac{u}{x-1}\frac{\partial}{\partial x}$ on X. Since for any non-empty input $u \in \mathcal{U}_{pc}$ the initial point $x_0 \notin X(f)$ (see Example 3.2 on the details for $u = 1$) and since $\mathcal{U}_{pc}(x_0) \subseteq \mathcal{U}_{pc}$, we know that $\mathcal{U}_{pc}(x_0) = \{e\}$.*

The following example shows the relevance of rational systems for modeling of biochemical processes.

Example 3.4 *Consider a biochemical reaction system modeling the initial part of the process of glycolysis in yeast. This example is borrowed from the book [32, Ex. 5.1]. The original model is published in [25]. The formulation of the biochemical system is based on the formalism of M. Feinberg, in particular on the paper [18].*

The state components are defined as the concentrations of the chemical species specified below.

x_1	*Gluc6P, glucose-6-phosphate,*
x_2	*Fruc6P, fructose-6-phosphate,*
x_3	*Fruc1,6P2, fructose-1,6-biphosphate,*
x_4	*ATP, adenosine-triphosphate,*
x_5	*ADP, adenosine-diphosphate,*
x_6	*AMP, adenosine-monophosphate,*
$x_{ex,1}$	*glucose,*
$x_{ex,2}$	*the chemical species produced by the last reaction.*

The variable $x_{ex,1}$ representing the glucose concentration outside the cell membrane is considered as the input. The variable $x_{ex,2}$ representing the outflow of the last reaction is considered as the output of the system.

The notation for the chemical complexes follows.

C_1	$x_4 + x_{ex,1} =$ *glucose + ATP,*
C_2	$x_1 + x_5 =$ *Gluc6P + ADP,*
C_3	$x_1 + x_4 =$ *Gluc6p + ATP,*
C_4	$x_5 =$ *ADP,*
C_5	$x_1 =$ *Gluc6P,*
C_6	$x_2 =$ *Fruc6P,*
C_7	$x_2 + x_4 =$ *Fruc6P + ATP,*
C_8	$x_3 + x_5 =$ *Fruc1,6P2 + ADP,*
C_9	$x_3 =$ *Fruc1,6P2,*
C_{10}	$x_4 =$ *ATP,*

C_{11}	$x_4 + x_6 = ATP + AMP$,
C_{12}	$2x_5 = 2ADP$,
C_{13}	$x_{ex,2} =$ *second external concentration.*

The reaction network represents the chemical reactions. It is defined as a graph whose vertices are represented by the chemical complexes defined above, and whose edges correspond to the existing reactions between the complexes. The set of edges is stated below.

$$\{(2,1),(4,3),(6,5),(5,6),(8,7),(13,9),(10,4),(4,10),(12,11),(11,12)\}.$$

To save space, we do not reproduce the reactions.
We define a matrix B relating the chemical species to the complexes. Let for a moment denote the variables $x_{ex,1}$ and $x_{ex,2}$ by x_7 and x_8, respectively. Then, $B = (B_{ik})_{i=1,\ldots,8,k=1,\ldots,13}$ where

$$B_{ik} = \begin{cases} n_k\ , & \textit{if chemical species } x_i \textit{ appears in complex } k \textit{ with multiplicity } n_k, \\ 0\ , & \textit{otherwise.} \end{cases}$$

Hence, the columns of the matrix B specify the structure of the complexes, i.e. members of reactionnet, and the rows specify the concentrations of chemical species in the system. From the matrix B one may compute the stoichiometric matrix which is not stated in this paper.
The reaction rates follow, specified per reaction by the reaction rate function.

$$\begin{aligned}
r_{2,1}(x) &= \frac{v_{max,1}x_4x_{ex,1}}{1 + x_4/k_{ATP,1} + x_{ex,1}/k_{glucosr,1} + (x_4/k_{ATP,1})(x_{ex,1}/k_{glucose,1})}, \\
r_{4,3}(x) &= k_2x_1x_4, \\
r_{6,5}(x) &= \frac{v^f_{max,3}}{k_{Gluc6P,3}}x_1[1 + \frac{x_1}{k_{Gluc6P,3}} + \frac{x_2}{k_{Fruc6P}}]^{-1}, \\
r_{5,6}(x) &= \frac{v^r_{max,3}}{k_{Fruc6P,3}}x_2[1 + \frac{x_1}{k_{Gluc6P,3}} + \frac{x_2}{k_{Fruc6P}}]^{-1}, \\
r_{8,7}(x) &= \frac{v_{max,4}x_22}{k_{Fruc6p,4}(1 + (x_4/x_6)2) + x_22}, \\
r_{13,9}(x) &= k_5x_3\ ,\quad r_{10,4}(x) = k_6x_5\ ,\quad r_{4,10}(x) = k_7x_4, \\
r_{12,11}(x) &= k_{8,f}x_4x_6\ ,\quad r_{11,12}(x) = k_{8,r}x_5^2.
\end{aligned}$$

The resulting biochemical system is then

$$\begin{aligned}
dx(t)/dt &= \sum_{(i,j)\in reactionnet} [B(i) - B(j)]r_{i,j}(x(t))u_{i,j}(t),\ x(t_0) = x_0, \\
y(t) &= h(x(t)) = r_{13,9}(x(t)) = k_5x_3(t).
\end{aligned}$$

where u represents the vector of enzyme concentrations which is also an input to the system. Note that the system is rational in the state though several reactions are only linear or polynomial in the state; and the system is rational in the input *variable $x_{ex,1}$, the glucose input, but linear in the enzyme input u. The output equation is linear in this case but in case the corresponding reaction rate is a rational function of the state then the output equation is rational too.*

One can prove that this rational system is positive, i.e. that the positive orthant is an invariant set. Also, for any state x from the positive orthant, the denominator of $r_{i,j}(x)$ is strictly positive.

Derivations at the switching time-points

Definition 3.5 *Let $(u)(\alpha,t) \in \mathcal{U}_{pc}(x_0)$. We denote the derivation of a real function $\varphi : \mathcal{U}_{pc}(x_0) \to \mathbb{R}$ at the switching time-point T_u of the input $(u)(\alpha,t)$ (the input is defined on the time domain $[0,T_u) \cup [T_u, T_u+t)$) as*

$$(D_\alpha\varphi)(u) = \frac{d}{dt}\varphi((u)(\alpha,t))|_{t=0+}.$$

Let $\varphi : \mathcal{U}_{pc}(x_0) \to \mathbb{R}$ be a real function and let $u \in \mathcal{U}_{pc}(x_0)$. We define the function $\widehat{\varphi}_u(t) = \varphi(u_{[0,t)})$ for $t \in [0,T_u)$. If $u = (\alpha_1,t_1)\dots(\alpha_k,t_k)$ and if $t \in [\sum_{i=1}^n t_i, \sum_{i=1}^{n+1} t_i), n+1 \le k$, then $\widehat{\varphi}_u(t) = \varphi((\alpha_1,t_1)\dots(\alpha_n, t-\sum_{i=1}^n t_i))$. The derivation $(D_\alpha\varphi)(u)$ of a real function φ at the switching time-point T_u of an input $(u)(\alpha,t)$ is well-defined if the function $\widehat{\varphi}_{(u)(\alpha,t)}(\hat{t}) = \varphi((u)(\alpha,\hat{t}))$, $\hat{t} \in [T_u, T_u+t)$ is differentiable at T_u+.

In the definition below, we define the set $\widetilde{\mathcal{U}_{pc}}$ of admissible inputs.

Definition 3.6 *We define the set of admissible inputs $\widetilde{\mathcal{U}_{pc}}$ as a subset of the space of input functions $\mathcal{U}_{pc}(x_0)$ such that the input functions from $\widetilde{\mathcal{U}_{pc}}$ have finitely many switching time points. Hence, $u = (u_1,t_1)\dots(u_k,t_k) \in \widetilde{\mathcal{U}_{pc}}$ if $u \in \mathcal{U}_{pc}(x_0)$ and $k < \infty$.*

Remark 3.7 *Obviously, the empty input e belongs to $\widetilde{\mathcal{U}_{pc}}$.*

Note that for every $u \in \widetilde{\mathcal{U}_{pc}}$ and for every $t < T_u$ the input $u_t \in \widetilde{\mathcal{U}_{pc}}$. The same is true for the inputs from $\mathcal{U}_{pc}(x_0)$.

We say that the map $\varphi : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$ is *smooth*, or that it is a $\mathcal{C}^\infty$*- map*, if the derivations $D_{\alpha_1}\dots D_{\alpha_i}\varphi$ are well-defined on $\widetilde{\mathcal{U}_{pc}}$ for every $i \in \mathbb{N}$ and $\alpha_j \in U, j = 1,2,\dots,i$.

To simplify the notation, the derivation $D_{\alpha_1}\dots D_{\alpha_i}\varphi$ can be rewritten as $D_\alpha\varphi$ where α is the multiindex $\alpha = (\alpha_1,\dots,\alpha_i)$.

We say that the function $\varphi : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$ is *analytic at the switching time points* of the inputs from $\widetilde{\mathcal{U}_{pc}}$ if for every input $u = (u_1,t_1)\dots(u_k,t_k) \in \widetilde{\mathcal{U}_{pc}}$ the function

$$\varphi_{u_1,\dots,u_k}(t_1,\dots,t_k) = \varphi((u_1,t_1)\dots(u_k,t_k))$$

is analytic, i.e. we can write $\varphi_{u_1,\dots,u_k}$ in the form of convergent formal power series in k indeterminates. Note that obviously the functions φ analytic at the switching time points are such that for every $(u)(\alpha,0)(v) \in \widetilde{\mathcal{U}_{pc}}$

$$\varphi((u)(\alpha,0)(v) = \varphi((u)(v))). \tag{2}$$

Theorem 3.8 *The ring of convergent formal power series over $\mathbb{R}$ in finitely many indeterminates is an integral domain.*

Proof: According to Theorem 1 from [51, Vol.1, Ch.7] the algebra of formal power series over $\mathbb{R}$ in finitely many indeterminates is an integral domain. Then the subring of convergent formal power series of an integral domain is also an integral domain. □

Definition 3.9 *We denote the set of real functions $\varphi : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$ which are analytic at the switching time points of the inputs from $\widetilde{\mathcal{U}_{pc}}$ by $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. We refer to the elements of $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ as to the* analytic functions *on $\widetilde{\mathcal{U}_{pc}}$.*

Theorem 3.10 *The set of analytic functions $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ is an integral domain.*

Proof: Because $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ is a subalgebra of $\mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ (the set of functions which are analytic at the switching time points of piecewise constant inputs from $\mathcal{U}_{pc}$), then $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ is an integral domain if $\mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ is an integral domain.

We prove that $\mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ is an integral domain by proving that if $fg = 0$ for $f, g \in \mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ then either $f = 0$ or $g = 0$.

Let $f, g \in \mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ be such that $fg = 0$. From the analyticity of f and g, we know that for arbitrary $u_1, \ldots, u_k \in U$ the functions $f_{u_1,\ldots,u_k}$ and $g_{u_1,\ldots,u_k}$ can be written in the form of convergent formal power series in k indeterminates. Consider an arbitrary $u = (u_1, t_1) \ldots (u_k, t_k) \in \mathcal{U}_{pc}$. Then $fg(u) = f(u)g(u) = 0$. Because, by Theorem 3.8, the ring of convergent formal power series in finitely many indeterminates is an integral domain, it follows that $f_{u_1,\ldots,u_k} g_{u_1,\ldots,u_k} = 0$ implies that either $f_{u_1,\ldots,u_k} = 0$ or $g_{u_1,\ldots,u_k} = 0$. Therefore $f(u) = f_{u_1,\ldots,u_k}(t_1, \ldots, t_k) = 0$ or $g(u) = g_{u_1,\ldots,u_k}(t_1, \ldots, t_k) = 0$.

Suppose that there exist inputs $u = (u_1, t_1^u) \ldots (u_k, t_k^u) \in \mathcal{U}_{pc}$ and $v = (v_1, t_1^v) \ldots (v_k, t_k^v) \in \mathcal{U}_{pc}$ such that

(i) $fg((u)(v)) = f((u)(v))g((u)(v)) = 0$,

(ii) $f(u) \neq 0$ and $g(v) \neq 0$.

Because $f, g \in \mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$, we get from the previous paragraph of this proof that either $f_{u_1,\ldots,u_k,v_1,\ldots v_l} = 0$ or $g_{u_1,\ldots,u_k,v_1,\ldots v_l} = 0$ and therefore that either $f((u)(v)) = 0$ or $g((u)(v)) = 0$. From (ii) above and from the property (2) stated below Remark 3.7, it follows that

$$0 \neq f(u) = f_{u_1,\ldots,u_k}(t_1^u, \ldots, t_k^u) = f_{u_1,\ldots,u_k,v_1,\ldots v_l}(t_1^u, \ldots, t_k^u, 0, \ldots, 0)$$

and

$$0 \neq g(v) = g_{v_1,\dots,v_k}(t_1^v,\dots,t_k^v) = g_{u_1,\dots,u_k,v_1,\dots v_l}(0,\dots,0,t_1^v,\dots,t_k^v)$$

which means that both $f_{u_1,\dots,u_k,v_1,\dots v_l}$ and $g_{u_1,\dots,u_k,v_1,\dots v_l}$ are non-zero convergent formal power series in $k+l$ indeterminates which contradicts that either $f_{u_1,\dots,u_k,v_1,\dots v_l} = 0$ or $g_{u_1,\dots,u_k,v_1,\dots v_l} = 0$. Therefore the inputs u and v satisfying (i), (ii) do not exist. □

Corollary 3.11 *Because the set $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ of analytic functions is an integral domain, we can define the field $\mathcal{Q}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ of the quotients of elements of $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$.*

Trajectories of rational systems

The trajectories of rational systems are the same as the trajectories of rational vector fields defined in Definition 2.4. However, in the case of rational systems we emphasize the effect of inputs on trajectories. The change of rational vector field and subsequently the change of trajectory is caused by applying a different input to the system. To indicate this dependency of inputs and trajectories we introduce the following notation.

Note that the solution of a rational system $\Sigma = (X, f, h, x_0)$ is a map analytic at the switching time points of the input $u \in \widetilde{\mathcal{U}_{pc}}$ as it is defined in Definition 3.9.

Definition 3.12 *Let $\Sigma = (X, f, h, x_0)$ be a rational system and let A denote the algebra of polynomial functions on X. We define the input-to-state map $\tau : \widetilde{\mathcal{U}_{pc}} \to X$ as the map $\tau(u_t) = x(t, x_0, u)$ for $u \in \widetilde{\mathcal{U}_{pc}}$ and $t \in [0, T_u)$. The map τ^* determined by τ is defined as $\tau^* : A \to \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ such that $\tau^*(\varphi) = \varphi \circ \tau$ for all $\varphi \in A$.*

The image of an input $u \in \widetilde{\mathcal{U}_{pc}}$ by the map τ defined as above is the corresponding trajectory of the system Σ.

To simplify further reference we state some properties of the map τ^* in Proposition 3.13. The proof of this proposition is omitted because it directly follows from the definition of τ^*.

Proposition 3.13 *Let $\Sigma = (X, f, h, x_0)$ be a rational system, let A be the finitely generated algebra of polynomials on X, i.e. there exist $\varphi_1, \dots, \varphi_k \in A$ such that $A = \mathbb{R}[\varphi_1, \dots, \varphi_k]$, and let Q denote the field of rational functions on X. Then the map $\tau^* : A \to \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ defined in Definition 3.12 is an homomorphism and $\tau^*(\mathbb{R}[\varphi_1, \dots, \varphi_k]) = \mathbb{R}[\tau^*\varphi_1, \dots, \tau^*\varphi_k]$. Moreover, the map $\widehat{\tau^*} : A/\mathrm{Ker}\ \tau^* \to \mathbb{R}[\tau^*\varphi_1, \dots, \tau^*\varphi_k]$, defined as $\widehat{\tau^*}([\varphi]) = \tau^*\varphi$ for every $\varphi \in A$, is an isomorphism. The map $\widehat{\tau^*}$ can be extended to an isomorphism of the fields $\mathcal{Q}(A/\mathrm{Ker}\ \tau^*)$ and $\mathbb{R}(\tau^*\varphi_1, \dots, \tau^*\varphi_k)$.*

The dimension of the state space

The state spaces which we consider in the previously introduced concept of rational systems are irreducible real affine varieties. Therefore, to determine the dimension of a rational system, and thus the dimension of its state space, we need to determine the dimension of a variety.

The dimension of an irreducible affine variety is defined as the degree of the affine Hilbert polynomial of the corresponding ideal of polynomials which vanish on the studied variety. There are other ways to describe this dimension and we follow one of them relating the Hilbert polynomial approach with the number of algebraically independent polynomials on a variety.

According to Theorem 2 from [13, Section 9.5], the dimension of an irreducible real affine variety X equals the maximal number of polynomials on X which are algebraically independent over $\mathbb{R}$. Since the rational functions on X are defined as the field of quotients of polynomials on X, we derive by considering the definition of the dimension of a variety by polynomials and by Definition 2.7(b.1) that the dimension of an irreducible real affine variety X equals the maximal number of rational functions on X which are algebraically independent over $\mathbb{R}$. The complete proof of this statement can be found in [13, Section 9.5, Theorem 6].

We use the characterization of the dimension of an irreducible real affine variety by the transcendence degree of the field of rational functions on X as its definition.

Definition 3.14 *Let X be an irreducible real affine variety and let Q denote the field of rational functions on X. Then* $\dim X = \operatorname{trdeg} Q$.

Controllability and observability of a rational system

The system is called controllable from an initial state if we can stear it to any state in the state space. We weaken this definition and we call a rational system controllable from an initial state if the set of all states which can be reached by applying a suitable input fuction to the system at the initial state is almost the full state space. This means that, since the state spaces of rational systems are just varieties, the smallest variety containing this set is already the state space of the considered system.

Definition 3.15 *The rational system Σ is said to be* controllable *(or alternatively* rationally reachable*)* from the initial state $x_0 \in X$ *if the reachable set from x_0,*

$$\mathcal{R}(x_0) = \{x(T_u, x_0, u) \in X | u \in \mathcal{U}_{pc}(x_0), u : [0, T_u) \longrightarrow U\},$$

is Z-dense in X. The rational system Σ is called controllable *(or* rationally reachable*) if it is controllable from any $x_0 \in X_\Sigma$.*

Proposition 3.16 *Let $\Sigma = (X, f, h, x_0)$ be a rational system. Then the closure $Z - cl\mathcal{R}(x_0)$ of the reachable set $\mathcal{R}(x_0)$ in Zariski topology on X is an irreducible variety.*

Proof: Because a subalgebra of an integral domain is an integral domain and because by Theorem 3.10 $\mathcal{A}(\widetilde{\mathcal{U}_{pc} \to \mathbb{R}})$ is an integral domain, we get that the algebra $\mathbb{R}[\tau^*\varphi_1, \ldots, \tau^*\varphi_k] \subseteq \mathcal{A}(\widetilde{\mathcal{U}_{pc} \to \mathbb{R}})$ is an integral domain. Due to Proposition 3.13, the map $\widehat{\tau^*} : A/\text{Ker } \tau^* \to \mathbb{R}[\tau^*\varphi_1, \ldots, \tau^*\varphi_k]$ is an isomorphism. Therefore, since $\mathbb{R}[\tau^*\varphi_1, \ldots, \tau^*\varphi_k]$ is an integral domain and $\widehat{\tau^*}$ is an isomorphism, we obtain that Ker τ^* is a prime ideal.

As Ker $\tau^* = \{f \in A | f = 0 \text{ on } \mathcal{R}(x_0)\}$, the ideal I of polynomials from A defining the smallest variety containing $\mathcal{R}(x_0)$, which is the variety $Z - cl\mathcal{R}(x_0) = \{x \in X | f(x) = 0 \text{ for all } f \in I\}$, coincides with Ker τ^*, i.e. $I =$ Ker τ^*. Finally, because Ker τ^* and thus also I is a prime ideal, the variety $Z - cl\mathcal{R}(x_0)$ defined by I is irreducible. □

Observability of the system means that starting from two different initial states we cannot get the same output. We call such states distinguishable. In the case of rational systems, we consider state spaces which are varieties. The smallest system of functions on a variety distinguishing its points (distinguishing in the sense that if $x \neq y$ then there exists f from the system of functions such that $f(x) \neq f(y)$) is the algebra of all polynomials defined on the considered variety. Therefore, the smallest system of functions distinguishing points and containing rational functions is given as the system of fractions of polynomials, i.e. all rational functions on a variety. Then the natural way how to define observability for rational systems is to define it as a property of getting all rational functions just by applying rational vector fields to all components of an output function.

Definition 3.17 *Let $\Sigma = (X, f = \{f_\alpha | \alpha \in U\}, h)$ be a rational system and let Q denote the field of rational functions on X. The* observation algebra *$A_{obs}(\Sigma)$ of Σ is the smallest subalgebra of the field Q containing all components $h_i, i = 1, \ldots, r$ of h, and closed with respect to the derivations given by rational vector fields $f_\alpha, \alpha \in U$. The* observation field *$Q_{obs}(\Sigma)$ of the system Σ is the field of quotients of $A_{obs}(\Sigma)$. The rational system Σ is called* observable *if $Q_{obs}(\Sigma) = Q$.*

Note that the observation algebra of a rational system is an integral domain because it is a subalgebra of an integral domain. Therefore the observation field of a rational system is well-defined. The observation field $Q_{obs}(\Sigma)$ is also closed with respect to the derivations given by rational vector fields $f_\alpha, \alpha \in U$.

Proposition 3.18 *([5], Proposition 1) For a rational system Σ, $Q_{obs}(\Sigma)$ is a finitely generated field extension of $\mathbb{R}$, i.e. there exist $\varphi_1, \ldots, \varphi_k \in Q_{obs}(\Sigma)$ such that $Q_{obs}(\Sigma) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$.*

Definition 3.19 *We call a rational system* canonical *if it is both observable and controllable.*

Example 3.20 *Consider a rational system $\Sigma = (X, f, h)$ where*

$$\begin{aligned} X &= \mathbb{R}^2, \\ f &= \{f_\alpha | \alpha \in \mathbb{R}\} = \{\frac{1}{x_1 x_2}\frac{\partial}{\partial x_1} + \alpha \frac{\partial}{\partial x_1} | \alpha \in \mathbb{R}\}, \\ h &= x_1^2. \end{aligned}$$

We will determine whether Σ is controllable and/or observable.

To check observability, we first need to compute the observation algebra $A_{obs}(\Sigma)$. We know that it needs to contain

$$\begin{aligned} h &= x_1^2, \\ f_\alpha h &= \frac{1}{x_1 x_2}\frac{\partial}{\partial x_1}(x_1^2) + \alpha\frac{\partial}{\partial x_1}(x_1^2) = \frac{2}{x_2} + 2\alpha x_1, \\ f_\beta f_\alpha h &= \frac{2\alpha}{x_1 x_2} + 2\alpha\beta, \\ f_\gamma(\frac{1}{x_1 x_2}) &= \frac{-1}{x_1^3 x_2^2} - \frac{\gamma}{x_1^2 x_2}, \quad f_\gamma(\frac{1}{x_1^2 x_2}) = \frac{-2}{x_1^4 x_2^2} - \frac{2\gamma}{x_1^3 x_2}, \quad \ldots \end{aligned}$$

After calculating several more derivations we observe that $A_{obs}(\Sigma)$ is generated by the set $\{x_1, \frac{1}{x_2}, \frac{1}{x_1 x_2}, \frac{1}{x_1^2 x_2}, \frac{1}{x_1^3 x_2}, \ldots\}$. Thus the observation algebra of Σ is not finitely generated. To compute the observation field $Q_{obs}(\Sigma)$, we construct the field of quotients of $A_{obs}(\Sigma) = \mathbb{R}[x_1, \frac{1}{x_2}, \frac{1}{x_1 x_2}, \frac{1}{x_1^2 x_2}, \frac{1}{x_1^3 x_2}, \ldots]$. We obtain $Q_{obs}(\Sigma) = \mathbb{R}(x_1, x_2)$ which is the finitely generated field of all rational functions on X. Therefore the system $\Sigma = (X, f, h)$ is observable.

To check controllability, we study reachable sets of Σ. The reachable set of an initial point $x_0 = (x_0^1, x_0^2) \in X_\Sigma$ for the rational system Σ is a subset of the line $x = x_0^2$ which is not Z-dense in $\mathbb{R}^2$. So, Σ is not controllable from any $x_0 \in X_\Sigma$ and therefore Σ is not controllable.

The formal definition of the rational realization problem will be stated in the next section. Before that we explain why we define objects as observation algebra and field for response maps as well.

In realization theory one is given a response map (or an input/output map) and then one is supposed to find a system within a certain class of systems which corresponds to this map. This map is therefore representing the unknown system. For this reason it is useful to define objects as observation algebra and observation field based on a response map rather than on the system realizing this map.

Let us consider a response map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$. We assume that the components $p_i : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$ of the map $p = (p_1, \ldots, p_r)$ are analytic in the sense described in the subsection on rational systems of this section. Thus we assume that $p_i \in \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ for $i = 1, \ldots, r$. This assumption on analyticity of p

allows us to define the observation field of p properly since there we need the property that a subalgebra of the class of components of considered response maps is an integral domain which is true for the response maps with the components in $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. Anyway, for well-definedness of the observation algebra of p it is sufficient to assume that the components of p are smooth in the sense described in the subsection on rational systems of this section.

Definition 3.21 *We call a map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$* analytic *if its components $p_i : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$, $i = 1, \dots, r$ are such that $p_i \in \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$.*

Definition 3.22 *Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be an analytic map. The observation algebra $A_{obs}(p)$ of p is the smallest subalgebra of the algebra $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ which contains the components $p_i, i = 1, \dots, r$ of p, and which is closed with respect to the derivations $D_\alpha, \alpha \in U$. The observation field $Q_{obs}(p)$ of p is the field of quotients of $A_{obs}(p)$.*

We recall again that $Q_{obs}(p)$ is well-defined only if $A_{obs}(p)$ is an integral domain which is the case for p being an analytic map.

4 Rational realizations

Response maps

In this paper we work with response maps rather than with input/output (I/O) maps. It is technically more convenient.

Usually the I/O maps are considered to be maps between spaces of functions mapping an input to an output. The elements of both input and output space are functions of time. If we map by such an I/O map an input at the fixed time, then we get certain output evaluated at this time point as an image. We call this map, which describes the outputs immediately after applying finite parts of the inputs, a response map.

Since the input space $\widetilde{\mathcal{U}_{pc}}$ we consider has a property of containing all restrictions of any input to shorter time domains starting at zero, we can map by an I/O map the inputs received by shortening time domain of an original input to get the values of an output function corresponding to the original input evaluated at the end time points of time domains of considered inputs. The output function corresponding to an input by an I/O map can be then constructed as the map which maps the value t of time (time point) to the image of the restriction of the input to the time interval $[0, t]$ by the I/O map. Therefore the "I/O maps" (we mean here already response maps) in our case can be considered as the maps between the input space $\widetilde{\mathcal{U}_{pc}}$ and the set of values $\mathbb{R}^r$ of output functions.

In the previous section we have defined an observation algebra and an observation field of a response map. These objects are used to solve the problem of realization of a response map by a rational system. Therefore we have to assume the smoothness of a response map for the observation algebra to be well-defined and we have to assume that a response map is analytic at the switching time points for the observation field to be well-defined.

The response maps analytic at the switching time points are also smooth with respect to D_α derivations. Therefore, in the rest of the paper we consider response maps $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ to be analytic (see Definition 3.21).

Problem formulation

Consider an analytic response map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$. A rational system which for each input gives us the same output as the map p is called a *rational realization of* p (a rational system *realizing* p). The realization problem for rational systems can be then understood as the problem of finding such a rational system with an initial state for a given map.

Problem 4.1 *Consider an analytic map* $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$. *The realization problem of rational systems consists of determining a rational system* $\Sigma = (X, f, h)$ *and an initial state* $x_0 \in X(f)$ *such that*

$$p(u_t) = h(x(t, x_0, u)) \text{ for all } u \in \widetilde{\mathcal{U}_{pc}} \text{ and } t \in [0, T_u],$$

for the considered analytic response map p.

Existence of rational realizations

In this subsection we provide sufficient and necessary conditions for an analytic response map to be realizable by a rational system. The analyticity of a map p, which we already assume, is necessary for the existence of a rational system realizing p. The question of realizability of response maps by a polynomial system is treated in [6, Theorem 2]. The proof of that theorem and the proof of Proposition 4.8 have the same structure.

Theorem 4.2 *If* F *is a finitely generated field containing* $\mathbb{R}$, *then every subfield* G *of* F *containing* $\mathbb{R}$ *is finitely generated.*

Proof: This is a consequence of a more general theorem stating that if F is a finitely generated extension of a field K, then every subextension G of F is finitely generated. See [11, Chapter V, Section 14.7, Corollary 3] for the proof. □

The following lemma can be found in [6] stated for polynomial systems.

Lemma 4.3 *Let* $\Sigma = (X, f, h, x_0)$ *be a rational system and let* $\tau : \widetilde{\mathcal{U}_{pc}} \to X$ *be as in Definition 3.12. Then for any* φ *from the algebra* A *of polynomials on* X *it holds that* $D_\alpha(\varphi \circ \tau) = (f_\alpha \varphi) \circ \tau$.

Proof: Let $u \in \widetilde{\mathcal{U}_{pc}}$ and let $t \in [0, T_u]$. Because $\widetilde{\mathcal{U}_{pc}} \subseteq \mathcal{U}_{pc}(x_0)$, the trajectory τ determined by $u \in \widetilde{\mathcal{U}_{pc}}$ is well-defined. We directly compute the derivation D_α of $\varphi \circ \tau$ for arbitrary $\varphi \in A$ and $\alpha \in U$. By Definition 3.5,

$$D_\alpha(\varphi \circ \tau)(u_t) = \frac{d}{ds}(\varphi \circ \tau)((u_t)(\alpha, s))|_{s=0+} = \frac{d}{ds}\varphi(\tau((u_t)(\alpha, s)))|_{s=0+}.$$

Since $\tau((u_t)(\alpha, s))$ is defined as a trajectory of a rational vector field f specified by an input $(u_t)(\alpha, s)$, we get that

$$D_\alpha(\varphi \circ \tau)(u_t) = \frac{d}{ds}\varphi(x(t+s, x_0, (u_t)(\alpha, s)))|_{s=0+}.$$

Consequently, by Definition 2.4,

$$D_\alpha(\varphi \circ \tau)(u_t) = (f_\alpha \varphi)(x(t+s, x_0, (u_t)(\alpha, s)))|_{s=0+}$$

and finally, by the continuity of rational function $f_\alpha \varphi$ and by the properties of trajectory x, we get that

$$D_\alpha(\varphi \circ \tau)(u_t) = (f_\alpha \varphi)(\tau(u_t)).$$

Therefore $D_\alpha(\varphi \circ \tau) = (f_\alpha \varphi) \circ \tau$ for any $u \in \widetilde{\mathcal{U}_{pc}}$. □

Proposition 4.4 *Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be an analytic map realizable by a rational system $\Sigma = (X, f, h, x_0)$. Let $\tau : \widetilde{\mathcal{U}_{pc}} \to X$ be as in Definition 3.12. Then the map $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$ defined as $\tau^*_{ext}\varphi = \varphi \circ \tau$ for every $\varphi \in \mathcal{A}_{obs}(\Sigma)$ is a well-defined surjective homomorphism, i.e. $\tau^*_{ext}(A_{obs}(\Sigma)) = A_{obs}(p)$.*

Proof: Note that τ^*_{ext} is defined in the same way as τ^* but on a different domain. See Definition 3.12 for details. Obviously, τ^*_{ext} is a homomorphism. We prove that τ^*_{ext} is well-defined and that it is surjective.

The observation algebras of a system Σ and of a map p are generated by the functions h_i, $f_{\alpha_1} \dots f_{\alpha_j} h_i$ and p_i, $D_{\alpha_1} \dots D_{\alpha_j} p_i$, respectively, such that $j \in \mathbb{N}$, $\alpha_1, \dots, \alpha_j \in U$, and $i \in \{1 \dots r\}$ (see Definitions 3.17 and 3.22). Since τ^*_{ext} is a homomorphism, to prove that $\tau^*_{ext}(A_{obs}(\Sigma)) = A_{obs}(p)$, it is sufficient to prove that the generators of $A_{obs}(\Sigma)$ and $A_{obs}(p)$ are mapped to each other by τ^*_{ext}. To prove that τ^*_{ext} is well-defined it is sufficient to prove that τ^*_{ext} is well-defined for the generators of the algebra $A_{obs}(\Sigma)$.

Since Σ is a rational realization of p, we know that $p = h \circ \tau$ and that p is well-defined. Thus $p = \tau^*_{ext} h = h \circ \tau = \frac{h_{num}}{h_{den}} \circ \tau = \frac{h_{num} \circ \tau}{h_{den} \circ \tau}$, where $h_{num}, h_{den} \in A$, is well-defined. Hence τ^*_{ext} is well-defined on h and $\tau^*_{ext} h = p$. For a rational vector field $f_\alpha \in f$, it holds that $f_\alpha h \in Q$. Moreover,

$$(f_\alpha h) \circ \tau = (f_\alpha \frac{h_{num}}{h_{den}}) \circ \tau = \frac{(f_\alpha h_{num} \circ \tau)(h_{den} \circ \tau) - (f_\alpha h_{den} \circ \tau)(h_{num} \circ \tau)}{(h_{den} \circ \tau)^2},$$

and by Lemma 4.3

$$(f_\alpha h) \circ \tau = \frac{D_\alpha(h_{num} \circ \tau)(h_{den} \circ \tau) - D_\alpha(h_{den} \circ \tau)(h_{num} \circ \tau)}{(h_{den} \circ \tau)^2}.$$

Therefore $(f_\alpha h) \circ \tau = D_\alpha(\frac{h_{num}}{h_{den}} \circ \tau) = D_\alpha(h \circ \tau) = D_\alpha(p)$ and finally we get that

$$\tau^*_{ext}(f_\alpha h) = D_\alpha(p). \tag{3}$$

Because p is an analytic response map, the derivations D_α of p are well-defined. Since p is realized by a rational system Σ, we know from (3) that τ^*_{ext} is well-defined on the generators of $A_{obs}(\Sigma)$, and that these generators are mapped by τ^*_{ext} to the generators $D_\alpha(p)$ of $A_{obs}(p)$.

Finally, because τ^*_{ext} is well-defined for all generators of $A_{obs}(\Sigma)$ and because they are mapped to the set of all generators of $A_{obs}(p)$, we can conclude that $\tau^*_{ext}(A_{obs}(\Sigma)) = A_{obs}(p)$. □

Corollary 4.5 *Since* $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$ *is, due to Proposition 4.4, an onto homomorphism, we get that* trdeg $A_{obs}(p) \leq$ trdeg $A_{obs}(\Sigma)$. *Moreover, even* trdeg $Q_{obs}(p) \leq$ trdeg $Q_{obs}(\Sigma)$.

In the proposition below we state necessary conditions for an analytic response map to be realizable by a rational system.

Proposition 4.6 *Let* $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ *be an analytic map realizable by a rational system* $\Sigma = (X, f, h, x_0)$. *Let* $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$ *be as in Proposition 4.4. Then*

(i) $Q_{obs}(p) = \widehat{\tau^*}(\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}))$, *where* $\widehat{\tau^*} : A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext} \to A_{obs}(p)$ *is an isomorphism derived from the map* $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$,

(ii) $Q_{obs}(p)$ *is finitely generated.*

Proof:
(i) From Proposition 4.4 we know that the map $\tau^*_{ext} : A_{obs}(\Sigma) \to A_{obs}(p)$ is a homomorphism which is surjective, but not necessarily injective. Then the map

$$\widehat{\tau^*} : A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext} \to A_{obs}(p)$$

defined as $\widehat{\tau^*}([\varphi]) = \tau^*_{ext}(\varphi)$ for every $\varphi \in A_{obs}(\Sigma)$, is an isomorphism. Note that $[\varphi_1] = [\varphi_2]$ if and only if $\varphi_1 - \varphi_2 \in \text{Ker } \tau^*_{ext}$. Since the algebras $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ and $A_{obs}(p)$ are integral domains, we can construct the fields of fractions of $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ and $A_{obs}(p)$.

So, as $\widehat{\tau^*} : A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext} \to A_{obs}(p)$ is an isomorphism, we can extend it to the isomorphism $\widehat{\tau^*}$ of the fields $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ and $Q_{obs}(p)$. Therefore, we get that $Q_{obs}(p) = \widehat{\tau^*}(\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}))$.

(ii) The map $\iota : Q_{obs}(\Sigma) = \mathcal{Q}(A_{obs}(\Sigma)) \to \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ defined naturally as $\iota(\frac{f_{num}}{f_{den}}) = \frac{[f_{num}]}{[f_{den}]}$ for $f_{num}, f_{den} \in A_{obs}(\Sigma)$ such that $f_{den} \neq 0$, is a surjective homomorphism. By Proposition 3.18 (or in another way, by Theorem 4.2 and by the fact that $Q_{obs}(\Sigma)$ is a subfield of a finitely generated field Q of rational functions on X), $Q_{obs}(\Sigma)$ is finitely generated. Therefore also a homomorphic image $\iota(Q_{obs}(\Sigma))$ is finitely generated and thus the field $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$ is finitely generated. Let $\varphi_1, \ldots, \varphi_k$ be the generators of $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$, i.e.

$$\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) = \mathbb{R}(\varphi_1, \ldots, \varphi_k). \tag{4}$$

From (i) we know that $Q_{obs}(p) = \widehat{\tau^*}(\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}))$ and that $\widehat{\tau^*}$ is an isomorphism. Then, by (4), $Q_{obs}(p) = \widehat{\tau^*}(\mathbb{R}(\varphi_1, \ldots, \varphi_k)) = \mathbb{R}(\widehat{\tau^*}\varphi_1, \ldots, \widehat{\tau^*}\varphi_k)$. Thus $Q_{obs}(p)$ is finitely generated. □

In the next proposition we state that we can choose the generators of the observation field $Q_{obs}(p)$ of an analytic map p from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. This allows us to reformulate the necessary condition for an analytic map p to be realizable by a rational system which is stated as condition (ii) in the proposition above as: $Q_{obs}(p)$ is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$.

Proposition 4.7 *Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be an analytic map. The observation field $Q_{obs}(p)$ is finitely generated if and only if it is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$, i.e. there exist finitely many $\varphi_1, \ldots, \varphi_k \in \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ such that $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$.*

Proof:
($\Leftarrow$) Let $Q_{obs}(p)$ be finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. Then it is obviously finitely generated.

($\Rightarrow$) Let $Q_{obs}(p)$ be finitely generated. Then there exist $\varphi_1, \ldots, \varphi_k \in Q_{obs}(p)$ such that $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$. As $\varphi_i \in Q_{obs}(p)$, $i = 1, \ldots, k$ we know that $\varphi_i = \frac{\varphi_{i,num}}{\varphi_{i,den}}$ where $\varphi_{i,num}, \varphi_{i,den} \in A_{obs}(p)$ for $i = 1, \ldots, k$. We define the field

$$F = \mathbb{R}(\varphi_{1,num}, \varphi_{1,den}, \ldots, \varphi_{k,num}, \varphi_{k,den}).$$

The map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ is analytic. Therefore all components $p_i, i = 1, \ldots, r$ of p are analytic, i.e. $p_i \in \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ for $i = 1, \ldots, r$. Because the algebra $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ is closed with respect to D_α derivations, we get that arbitrary D_α derivation of p_i belongs to $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. Since the observation algebra $A_{obs}(p)$ of p is generated by $p_i, D_\alpha p_i, i = 1, \ldots, r, \alpha = (\alpha_1, \ldots, \alpha_j), \alpha_k \in U, k = 1, \ldots, j, j \in \mathbb{N}$ and since $\varphi_{i,num}, \varphi_{i,den} \in A_{obs}(p)$ for $i = 1, \ldots, k$, and $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ is an algeba, it follows that

$$\varphi_{i,num}, \varphi_{i,den} \in \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R}) \text{ for } i = 1, \ldots, k, \tag{5}$$

and

$$\mathbb{R}[\varphi_{1,num}, \varphi_{1,den}, \ldots, \varphi_{k,num}, \varphi_{k,den}] \subseteq A_{obs}(p). \tag{6}$$

From (5) we know that the field F is generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. Therefore to prove that $Q_{obs}(p)$ is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ it is sufficient to prove that $F = Q_{obs}(p)$. Because from the definition of F it is obvious that $F \supseteq Q_{obs}(p)$ and because from (6) we get by taking the quotients that $F \subseteq Q_{obs}(p)$, the field $Q_{obs}(p)$ is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. □

The following proposition provides a characterization of analytic maps which are realizable by a rational system. It specifies sufficient conditions for rational realizability of an analytic response map.

Proposition 4.8 *Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be an analytic map. If there exists a field $F \subseteq \mathcal{Q}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ such that*

(i) F is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$,

(ii) F is closed with respect to D_α derivations, i.e.

$$\forall i \in \mathbb{N},\ \forall \alpha_j \in U, j = 1, \ldots, i : D_{\alpha_1} \ldots D_{\alpha_i} F \subseteq F,$$

(iii) $Q_{obs}(p) \subseteq F$,

then p has a rational realization.

Proof: Consider an analytic map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$. We assume that there exists a field $F \subseteq \mathcal{Q}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ which is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$, closed with respect to D_α derivations, and containing $Q_{obs}(p)$.

Since F is assumed to be a field which is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$, we know that there exist finitely many analytic functions $\varphi_1, \ldots, \varphi_k : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$ such that $F = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$. We call these functions the generators of F.

Because F is assumed to be closed with respect to D_α derivations, any D_α derivation of a generator of F can be expressed as a rational combination of the generators $\varphi_1, \ldots, \varphi_k$ of F. Formally, for any $\alpha = (\alpha_1, \ldots, \alpha_j)$ such that $\alpha_1, \ldots, \alpha_j \in U, j \in \mathbb{N}$, and for every $\varphi_i, i = 1, \ldots, k$, there exists a rational function v_i^α with real coefficients in k variables such that

$$D_\alpha \varphi_i = v_i^\alpha(\varphi_1, \ldots, \varphi_k).$$

From the assumption that $Q_{obs}(p) \subseteq F$ we know that the components of the map $p = (p_1, \ldots, p_r)$ are the elements of F and thus they can be expressed as rational combinations of the generators of F, i.e. for every $j = 1, \ldots, r$,

$$p_j = w_j(\varphi_1, \ldots, \varphi_k),$$

where w_j is a rational function with real coefficients in k variables.

Consider a rational system $\Sigma = (X, f, h, x_0)$ such that

$$
\begin{aligned}
&X = \mathbb{R}^k,\\
&f_\alpha = \sum_{i=1}^{k} v_i^\alpha \frac{\partial}{\partial x_i}, \alpha \in U,\\
&h_j(x_1, \dots, x_k) = w_j(x_1, \dots, x_k), j = 1 \dots r,\\
&x_0 = (\varphi_1(e), \dots, \varphi_k(e)) \text{ where } e \text{ is an empty input, i.e. } T_e = 0.
\end{aligned}
$$

We will prove that $\Sigma = (X, f, h, x_0)$ is a rational realization of p.

First we have to prove that Σ is a reasonable rational system meaning that there exists a solution of Σ.

Let us define $\Psi(t) = (\varphi_1, \dots, \varphi_k)(u_t)$ for $u \in \widetilde{\mathcal{U}_{pc}}, t \in [0, T_u)$. It is well-defined because the functions $\varphi_i, i = 1, \dots, k$ are defined for every $u \in \widetilde{\mathcal{U}_{pc}}$ and moreover they are analytic at the switching time points of the inputs from $\widetilde{\mathcal{U}_{pc}}$. Note that $\Psi(0) = (\varphi_1, \dots, \varphi_k)(u_0)$ is also well-defined because $u_0 = e$ and because the empty input e belongs to $\widetilde{\mathcal{U}_{pc}}$ (see Remark 3.7) and the analytic functions $\varphi_1, \dots, \varphi_k : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$ are defined properly at e. We prove that for $u \in \widetilde{\mathcal{U}_{pc}}$ and $t \in [0, T_u)$, $\Psi(t) = x(t, x_0, u)$. Then the system Σ is solvable and its trajectories are described by Ψ.

Consider a constant input $u_t = (\alpha, t) \in \widetilde{\mathcal{U}_{pc}}$, $\alpha \in U, t \geq 0$. Hence, u_t is a constant input with value α on the time interval $[0, t)$ for $t \geq 0$. Then

$$
\begin{aligned}
\Psi(0) &= (\varphi_1, \dots, \varphi_k)(u_0) = (\varphi_1(e), \dots, \varphi_k(e)) = x_0 \text{ , and}\\
\frac{d}{dt}\Psi(t) &= \frac{d}{d\tau}\Psi(t+\tau)|_{\tau=0} = \frac{d}{d\tau}(\varphi_1(u_{t+\tau}), \dots, \varphi_k(u_{t+\tau}))|_{\tau=0+}.
\end{aligned}
$$

Because $(u_t)(\alpha, \tau) = u_{t+\tau}$ (this is a consequence of u_t being a constant input with the value α), and because $D_\alpha \varphi(u_t) = \frac{d}{d\tau}\varphi((u_t)(\alpha, \tau))|_{\tau=0+}$, the derivation $\frac{d}{dt}\Psi(t)$ equals $(D_\alpha\varphi_1(u_t), \dots, D_\alpha\varphi_k(u_t))$. Since D_α derivations of the functions $\varphi_i, i = 1, \dots, k$ can be rewritten as $v_i^\alpha(\varphi_1, \dots, \varphi_k)$, we get that $\frac{d}{dt}\Psi(t) = (v_1^\alpha(\varphi_1, \dots, \varphi_k)(u_t), \dots, v_k^\alpha(\varphi_1, \dots, \varphi_k)(u_t))$. Finally, because we defined $\Psi(t)$ as $\Psi(t) = (\varphi_1, \dots, \varphi_k)(u_t)$, we get that

$$
\frac{d}{dt}\Psi(t) = (v_1^\alpha(\Psi(t)), \dots, v_k^\alpha(\Psi(t))) \text{ , and } \Psi(0) = x_0.
$$

Hence $\Psi(t) = x(t, x_0, u)$ for a constant input $u \in \widetilde{\mathcal{U}_{pc}}, t \in [0, T_u)$. This proves that for a piecewise constant input $u = (\alpha_1, t_1) \dots (\alpha_j, t_j) \in \widetilde{\mathcal{U}_{pc}}$ and for $t \in [0, t_1)$, we have $\Psi(t) = x(t, x_0, u)$.

If we consider the input $u = (\alpha_1, t_1) \dots (\alpha_j, t_j) \in \widetilde{\mathcal{U}_{pc}}$ and if we take the time $t \in [t_1, t_1 + t_2)$ instead of $t \in [0, t_1)$, we get that $\Psi(t_1) = x(t_1, x_0, u)$ and $\frac{d}{dt}\Psi(t) = (v_1^{\alpha_2}(\Psi(t)), \dots, v_k^{\alpha_2}(\Psi(t)))$ with the same reasoning as in the previous paragraph. Hence, $\Psi(t) = x(t, \Psi(t_1), u) = x(t, x(t_1, x_0, u), u)$ for $t \in [t_1, t_1 + t_2)$. In the analogous way we can study the cases for $t \in [t_1 + t_2, t_1 + t_2 + t_3), \dots, t \in [t_1 + \dots + t_{i-1}, t_1 + \dots + t_i), \dots$.

Finally we get that $\Psi(t) = x(t, x_0, u)$ for an arbitrary $u \in \widetilde{\mathcal{U}_{pc}}$ and $t \in [0, T_u)$.

To prove that the rational system Σ is a realization of the response map p, we have to prove that $p(u_t) = h(x(t, x_0, u))$ for every $u \in \widetilde{\mathcal{U}_{pc}}$ and $t \in [0, T_u)$.

Consider an arbitrary $u \in \widetilde{\mathcal{U}_{pc}}$ and $t \in [0, T_u)$. Since every component p_i of p is expressed as w_i - rational combination of generators $\varphi_1, \ldots, \varphi_k$ of F, we can rewrite $p(u_t)$ as

$$p(u_t) = (p_1, \ldots, p_r)(u_t) = (w_1(\varphi_1, \ldots, \varphi_k), \ldots, w_r(\varphi_1, \ldots, \varphi_k))(u_t).$$

Then, by the definitions of $h_j, j = 1, \ldots, r$ and Ψ we get that

$$p(u_t) = (h_1(\varphi_1, \ldots, \varphi_k), \ldots, h_r(\varphi_1, \ldots, \varphi_k))(u_t) = (h_1(\Psi(t)), \ldots, h_r(\Psi(t))).$$

We have already proved that $\Psi(t) = x(t, x_0, u)$ for arbitrary $u \in \widetilde{\mathcal{U}_{pc}}$ and for all $t \in [0, T_u)$. Therefore we finally get that

$$p(u_t) = h(x(t, x_0, u)) \text{ for } u \in \widetilde{\mathcal{U}_{pc}}, t \in [0, T_u)$$

and thus we have proved that the rational system Σ is a rational realization of p. □

The main theorem of this section solving the problem of existence of rational realizations and characterization of all maps realizable by rational systems has the following form based on the last three propositions above.

Theorem 4.9 *An analytic map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ has a rational realization if and only if $Q_{obs}(p)$ is finitely generated.*

Proof: (⇒) See the part (ii) of Proposition 4.6 for this statement and the proof.
(⇐) From Proposition 4.8 we know that the existence of a finitely generated field $F \subseteq \mathcal{Q}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$, containing $Q_{obs}(p)$, and closed with respect to D_α derivations implies the rational realizability of p. Since we assume that $Q_{obs}(p)$ is finitely generated, we get by Proposition 4.7 that $Q_{obs}(p)$ is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. Therefore we can define the field F to be the observation field $Q_{obs}(p)$. Then, by following the steps of the proof of Proposition 4.8 for $F = Q_{obs}(p)$, we construct a rational realization of p. Hence, we can conclude that if $Q_{obs}(p)$ is finitely generated then p is realizable by a rational system. □

Example

We present an example to demonstrate how to construct a rational system realizing a given analytic response map. The example is motivated by an example stated in [6]. The procedure to construct the corresponding rational system is following the steps made in the proof of Proposition 4.8.

Example 4.10 *Let the input space $\widetilde{\mathcal{U}_{pc}}$ be the space of all piecewise constant functions with the values in U and with finitely many switching time points. Let $U = \mathbb{R}$. Then the admissible inputs are all piecewise constant inputs $u : \mathbb{R} \to \mathbb{R}$ with finitely many switching time points. We determine a rational system Σ realizing a map $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}$ which is defined as $p(u) = exp(\int_0^{T_u} \frac{u(s)}{(1+s)^2} ds)$.*

We could also consider a map p defined as $p(u_t) = exp(\int_0^{t} \frac{u(s)}{(1+s)^2} ds)$ for $t \in [0, T_u)$ (it is an I/O map) but due to the structure of $\widetilde{\mathcal{U}_{pc}}$ we get the same information by defining p as we did. See the subsection on response maps for more detailed explanation.

Firstly we compute D_α derivations of p. Let α_1 and α_2 be arbitrary real numbers and let $u \in \widetilde{\mathcal{U}_{pc}}$. Then

$$
\begin{aligned}
&(D_{\alpha_1} p)(u) = \\
&= \frac{d}{d\tau} p((u)(\alpha_1, \tau))|_{\tau=0} \\
&= \left[\frac{d}{d\tau} exp\left(\int_0^{T_u} \frac{u(s)}{(1+s)^2} ds + \int_{T_u}^{T_u+\tau} \frac{\alpha_1}{(1+s)^2} ds \right) \right]_{\tau=0} \\
&= \left[\frac{d}{d\tau} exp\left(\int_0^{T_u} \frac{u(s)}{(1+s)^2} ds \right) \right]_{\tau=0} exp\left(\int_{T_u}^{T_u+\tau} \frac{\alpha_1}{(1+s)^2} ds \right) \\
&\quad + exp\left(\int_0^{T_u} \frac{u(s)}{(1+s)^2} ds \right) \left[\frac{d}{d\tau} exp\left(\int_{T_u}^{T_u+\tau} \frac{\alpha_1}{(1+s)^2} ds \right) \right]_{\tau=0} \\
&= 0 + p(u)\alpha_1 \left[exp\left(\int_{T_u}^{T_u+\tau} \frac{\alpha_1}{(1+s)^2} ds \right) \right]_{\tau=0} \left[\frac{d}{d\tau} \int_{T_u}^{T_u+\tau} \frac{1}{(1+s)^2} ds \right]_{\tau=0} \\
&= \alpha_1 p(u) \frac{1}{(1+T_u)^2},
\end{aligned}
$$

$$
\begin{aligned}
&(D_{\alpha_2} D_{\alpha_1} p)(u) = \\
&= D_{\alpha_2}(\alpha_1 p(u) \frac{1}{(1+T_u)^2}) = \alpha_1 \frac{1}{(1+T_u)^2} D_{\alpha_2} p(u) + \alpha_1 p(u) D_{\alpha_2} \frac{1}{(1+T_u)^2} \\
&= \frac{\alpha_1 \alpha_2}{(1+T_u)^4} p(u) + \alpha_1 p(u) \left[\frac{d}{d\tau} \frac{1}{(1+T_u+\tau)^2} \right]_{\tau=0} \\
&= \alpha_1 \alpha_2 \frac{1}{(1+T_u)^4} p(u) + \alpha_1 p(u) \frac{-2}{(1+T_u)^3}.
\end{aligned}
$$

We could continue to compute the derivations $(D_{\alpha_i} \dots D_{\alpha_1} p)(u)$ for any $i \in \mathbb{N}, \alpha_j \in \mathbb{R}, j \in 1, \dots, i$. Anyway, if we define $\varphi_1(u) = p(u)$ and $\varphi_2(u) = 1 + T_u$, we can see that $(D_{\alpha_i} \dots D_{\alpha_1} p)(u) \in \mathbb{R}(\varphi_1(u), \varphi_2(u))$ for any $i \in \mathbb{N}$ and $\alpha_j \in \mathbb{R}, j \in 1, \dots, i$. Therefore, by Definition 3.22, $Q_{obs}(p) \subseteq \mathbb{R}(\varphi_1, \varphi_2)$ and consequently, by Theorem 4.2, $Q_{obs}(p)$ is finitely generated. Further, by Theorem 4.9, there exists a rational system realizing p.

We follow the proof of Proposition 4.8 to construct a rational system $\Sigma = (X, f, h, x_0)$ realizing p. We consider a field F such that $F = \mathbb{R}(\varphi_1, \varphi_2)$. It is finitely generated, it contains $Q_{obs}(p)$, and it is closed with respect to D_α derivations. The number of generators of F equals 2 which implies that the state space X can be taken as $\mathbb{R}^2$. To determine a family of rational vector fields $f = \{f_\alpha | \alpha \in \mathbb{R}\}$ we compute

$$v_1^\alpha(\varphi_1, \varphi_2) = D_\alpha \varphi_1 = D_\alpha p = \alpha \varphi_1 \frac{1}{\varphi_2^2} \quad \textit{and} \quad v_2^\alpha(\varphi_1, \varphi_2) = D_\alpha \varphi_2 = 1.$$

The last equality $D_\alpha \varphi_2 = 1$ is true because $D_\alpha \varphi_2(u) = \left[\frac{d}{d\tau}\varphi_2((u)(\alpha,\tau))\right]_{\tau=0} = \left[\frac{d}{d\tau}(1 + T_u + \tau)\right]_{\tau=0} = [1]_{\tau=0} = 1$ for any $u \in \widetilde{\mathcal{U}_{pc}}$. The output map h is determined by a map w such that $w(\varphi_1, \varphi_2) = p = \varphi_1$, and the initial point x_0 is given as $x_0 = (\varphi_1(e), \varphi_2(e))$ where e is an empty input.

Finally, the rational realization $\Sigma = (X, f, h, x_0)$ of p is given as

$$\begin{aligned}
&X = \mathbb{R}^2, \quad \alpha \in U, \\
&f_\alpha(x_1, x_2) = v_1^\alpha(x_1, x_2)\frac{\partial}{\partial x_1} + v_2^\alpha(x_1, x_2)\frac{\partial}{\partial x_2} = \alpha\frac{x_1}{x_2^2}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \\
&h(x_1, x_2) = w(x_1, x_2) = x_1, \\
&x_0 = (1, 1).
\end{aligned}$$

We can rewrite it in the system theoretic form as

$$\begin{aligned}
&\frac{dx(t)}{dt} = f(x(t), \alpha) = \begin{pmatrix} \alpha\frac{x_1(t)}{x_2^2(t)} \\ 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
&y(t) = h(x(t)) = x_1(t).
\end{aligned}$$

5 Canonical rational realizations

Recall that a canonical rational realization of an analytic response map is a rational realization which is both controllable and observable. See Definition 3.15, 3.17, and 3.19 for the details.

The systems which we call canonical are called minimal by Bartosiewicz. Hence Theorem 3 from [6] describing the conditions under which a map has a minimal polynomial realization corresponds to our Theorem 5.1 considering rational systems. The proofs are analogous.

Theorem 5.1 *Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be an analytic response map. The following statements are equivalent:*

a) p has an observable rational realization,

b) p has a canonical rational realization,

c) $Q_{obs}(p)$ is finitely generated.

Proof: $(a) \Rightarrow (b)$ Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be an analytic map. Consider a rational system $\Sigma = (X, f, h, x_0)$ realizing p, which is observable. Then Σ is canonical if it is controllable from the initial state x_0. In other words, if $Z - cl\mathcal{R}(x_0) = X$, then the observable rational system Σ is a canonical rational realization of p.

Let us denote the Zariski closure of $\mathcal{R}(x_0)$ by X', i.e. $X' = Z - cl\mathcal{R}(x_0)$. According to Proposition 3.16, the variety X' is an irreducible real affine variety. If $X' = X$, then the proof of the implication $(a) \Rightarrow (b)$ is complete as we have discussed in the previous paragraph.

Assume that $X' \neq X$. Then the rational realization Σ of p is observable but not controllable. Hence, to prove that there exists a canonical rational realization of p, we need to find another rational system Σ' having the required properties.

Because X' is an irreducible real affine variety, we can consider X' to be a state space of a rational system Σ'. We relate the algebras A and A' of polynomials on X and X', respectively. Let I be an ideal of polynomials from A which vanish on X'. Then the quotient ring A/I can be identified with A', i.e. $A/I \cong A'$. We denote the corresponding bijection by $\Psi : A/I \to A'$. This is one-to-one and onto mapping which preserves sums and products. Hence, we can think about the polynomials of A' as the polynomials of A in that sense that the polynomials of A which differ just outside X' are identified. Therefore, if we consider $\varphi' \in A'$ and $\varphi \in A$ such that $\Psi([\varphi]) = \varphi'$, we know that $\varphi \restriction_{X'} = \varphi'$. More details are in [13, Chapter 5.2].

The algebra A' of polynomials on X' is finitely generated. Thus there exist polynomials $\varphi'_1, \ldots, \varphi'_k$ from A' such that $A' = \mathbb{R}[\varphi'_1, \ldots, \varphi'_k]$. Moreover, A' is an integral domain and we can define the field Q' of rational functions on X' by getting fractions of polynomials of A'. Thus $Q' = \mathbb{R}(\varphi'_1, \ldots, \varphi'_k)$.

We define the rational system $\Sigma' = (X', f', h', x'_0)$ by considering the state space X' and by deriving the family of rational vector fields f', an output function h' and an initial state x'_0 from the rational system Σ.

Since X' was defined as $Z - cl\mathcal{R}(x_0)$, the initial state x_0 of Σ is in X'. Therefore we can define the initial state x'_0 of Σ' as $x'_0 = x_0$.

Recall that for $\varphi \in A$ such that $\Psi([\varphi]) = \varphi'$ it holds that $\varphi \restriction_{X'} = \varphi'$. We define the output function h' of Σ' as

$$\begin{aligned} h' &= (h'_1, \ldots, h'_r) = \left(\frac{h'_{1,num}}{h'_{1,den}}, \ldots, \frac{h'_{r,num}}{h'_{r,den}} \right) \\ &= \left(\frac{\Psi([h_{1,num}])}{\Psi([h_{1,den}])}, \ldots, \frac{\Psi([h_{r,num}])}{\Psi([h_{r,den}])} \right) = \left(\frac{h_{1,num} \restriction_{X'}}{h_{1,den} \restriction_{X'}}, \ldots, \frac{h_{r,num} \restriction_{X'}}{h_{r,den} \restriction_{X'}} \right) \\ &= (h_1 \restriction_{X'}, \ldots, h_r \restriction_{X'}) = h \restriction_{X'}, \end{aligned}$$

where $h'_{i,num}, h'_{i,den} \in A'$ and $h_{i,num}, h_{i,den} \in A$ for $i = 1, \ldots, r$. The output function h' is well-defined because $h'_{i,den} \neq 0$ on X' for $i = 1, \ldots, r$. This is due to the facts that $x_0 \in X(f)$, h is defined at x_0, $x_0 \in X'$ and that $h' = h \restriction_{X'}$.

We derive the rational vector fields $f' = \{f'_\alpha : Q' \to Q' | \alpha \in U\}$ in the following way. Let $q' = \frac{q'_{num}}{q'_{den}}$ where $q'_{num}, q'_{den} \in A'$ and $q'_{den} \neq 0$. Thus

$$f'_\alpha q' = f'_\alpha \frac{q'_{num}}{q'_{den}} = f'_\alpha \frac{\Psi([q_{num}])}{\Psi([q_{den}])}. \tag{7}$$

We define the relation between f'_α and f_α as

$$f'_\alpha \frac{\Psi([q_{num}])}{\Psi([q_{den}])} = \frac{\Psi([(f_\alpha \frac{q_{num}}{q_{den}})_{num}])}{\Psi([(f_\alpha \frac{q_{num}}{q_{den}})_{den}])}. \tag{8}$$

Because for $\varphi \in A$ such that $\Psi([\varphi]) = \varphi'$ it is true that $\varphi \restriction_{X'} = \varphi'$, we get from (8) that

$$f'_\alpha \frac{\Psi([q_{num}])}{\Psi([q_{den}])} = \frac{(f_\alpha \frac{q_{num}}{q_{den}})_{num} \restriction_{X'}}{(f_\alpha \frac{q_{num}}{q_{den}})_{den} \restriction_{X'}} = \left(f_\alpha \frac{q_{num}}{q_{den}} \right) \restriction_{X'} . \tag{9}$$

Finally, by (7) and (9)

$$f'_\alpha q' = (f_\alpha q) \restriction_{X'} .$$

The well-definedness of the rational vector fields $f'_\alpha, \alpha \in U$ follows from the fact that the fraction $\frac{q_{num}}{q_{den}} \restriction_{X'}$ is independent on the choice of representants $q_{num}, q_{den} \in A$ of classes $[q_{num}], [q_{den}] \in A/I$. To verify this we consider arbitrary $f, g \in I$. Recall that this means that f and g are zero maps on X' and therefore $q_{num} + f \in [q_{num}]$ and $q_{den} + g \in [q_{den}]$. The well-definedness of $f'_\alpha, \alpha \in U$ is then confirmed by the calculation

$$\frac{q_{num}}{q_{den}} \restriction_{X'} = \frac{q_{num} \restriction_{X'}}{q_{den} \restriction_{X'}} = \frac{(q_{num} + f) \restriction_{X'}}{(q_{den} + g) \restriction_{X'}} = \frac{q_{num} + f}{q_{den} + g} \restriction_{X'} .$$

We have derived a rational system $\Sigma' = (X', f', h', x'_0)$ from the rational system Σ such that

$$\begin{aligned}
X' &= Z - cl\mathcal{R}(x_0), \\
f' &= \{f'_\alpha | \alpha \in U\}, \text{ where } f'_\alpha q' = f_\alpha q \restriction_{X'} \text{ for each } \alpha \in U \text{ and for} \\
&\qquad\qquad q \in Q \text{ such that } \Psi([q]) = q', \\
h' &= h \restriction_{X'}, \\
x'_0 &= x_0.
\end{aligned}$$

The trajectory of Σ is a map $x : [0, T) \to X$ such that $\frac{d}{dt}(\varphi \circ x)(t) = (f\varphi)(x(t))$ and $\varphi(x(0)) = x_0$ for $t \in [0, T)$ and $\varphi \in A$. On the other hand, the trajectory of Σ' is a map $x' : [0, T') \to X'$ such that $\frac{d}{dt}(\varphi' \circ x')(t) = (f'\varphi')(x'(t))$ and $\varphi'(x'(0)) = x'_0$ for $t \in [0, T')$ and $\varphi' \in A'$. Because for every $\varphi' \in A'$ there exists $\varphi \in A$ such that $\varphi' = \Psi([\varphi])$ and because $f'_\alpha \Psi([\varphi]) = f_\alpha \varphi \restriction_{X'}$, we can compute $\frac{d}{dt}(\varphi' \circ x')(t) = (f'\varphi')(x'(t)) = (f'\Psi([\varphi]))(x'(t)) = (f\varphi) \restriction_{X'} (x'(t)) =$

$\frac{d}{dt}(\varphi \circ x') \restriction_{X'} (t)$. The state space X' was chosen to be equal to $Z - cl\mathcal{R}(x_0)$, therefore the trajectory x' cannot leave X' and hence

$$(f\varphi)(x'(t)) = \frac{d}{dt}(\varphi \circ x')(t). \tag{10}$$

Moreover, since $x'_0 = x_0$ we get that

$$\varphi'(x'(0)) = x'_0 = x_0 = \varphi(x(0)). \tag{11}$$

The equalities (10) and (11) imply that the trajectories x and x' of the systems Σ and Σ', respectively, are the same. Then the reachable sets of both systems coincide and since the state space X' is defined as $Z - cl\mathcal{R}(x_0)$, this leads us to the equality $X' = Z - cl\mathcal{R}(x'_0)$ meaning that the rational system $\Sigma' = (X', f', h', x'_0)$ is controllable.

From the equality of trajectories of Σ and Σ' and from the definition of the output function h' of Σ' as $h' = h \restriction_{X'}$, it follows that the system Σ' is a rational realization of p. We prove it in more detail. Let $u \in \widetilde{\mathcal{U}_{pc}}$ and let $t \in [0, T_u]$, then

$$\begin{aligned}
p(u_t) &= h(x(t, x_0, u)), \text{ because } \Sigma \text{ realizes } p, \\
&= (h \restriction_{X'})(x(t, x_0, u)), \text{ as } X' = Z - cl\mathcal{R}(x_0) \text{ and thus } x : [0, T) \to X', \\
&= (h \restriction_{X'})(x'(t, x'_0, u)), \text{ from the equalities of trajectories of } \Sigma, \Sigma', \\
&= h'(x'(t, x'_0, u)), \text{ by the definition of } h'.
\end{aligned}$$

Thus, the system Σ' is a controllable rational realization of p. If we prove that Σ' is even observable, the proof of the existence of a canonical rational realization of p will be complete.

Let us compute the observation field $Q_{obs}(\Sigma')$ of the system Σ'. Firstly, the observation algebra $A_{obs}(\Sigma')$ is the smallest algebra containing the elements $h'_i, f'_\alpha h'_i$ for $i = 1, \ldots, r$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$ such that $k \in \mathbb{N}, \alpha_j \in U, j = 1, \ldots k$. As $h'_i = h_i \restriction_{X'}$ and $f'_\alpha h'_i = f_\alpha h_i \restriction_{X'}$ for $i = 1, \ldots, k$ and $\alpha = (\alpha_1, \ldots, \alpha_k), \alpha_j \in U, k \in \mathbb{N}$, we derive the equation $A_{obs}(\Sigma') = A_{obs}(\Sigma) \restriction_{X'}$. This means that $\varphi = \frac{\varphi_{num}}{\varphi_{den}} \in A_{obs}(\Sigma)$ if and only if $\varphi' = \frac{\varphi'_{num}}{\varphi'_{den}} = \frac{\Psi([\varphi_{num}])}{\Psi([\varphi_{den}])} \in A_{obs}(\Sigma')$. Since $A_{obs}(\Sigma) \restriction_{X'}$ and $A_{obs}(\Sigma')$ are integral domains, we get by considering their quotient fields that $Q_{obs}(\Sigma') = \mathcal{Q}(A_{obs}(\Sigma) \restriction_{X'})$. Therefore, to prove that Σ' is observable, i.e. that $Q_{obs}(\Sigma') = Q' = \mathcal{Q}(A \restriction_{X'})$, we need to prove that

$$\mathcal{Q}(A_{obs}(\Sigma) \restriction_{X'}) = \mathcal{Q}(A \restriction_{X'}). \tag{12}$$

Generally $\mathcal{Q}(A_{obs}(\Sigma) \restriction_{X'}) \subseteq \mathcal{Q}(A \restriction_{X'})$. Hence, to prove (12) we prove that $\mathcal{Q}(A \restriction_{X'}) \subseteq \mathcal{Q}(A_{obs}(\Sigma) \restriction_{X'})$, i.e. if $f \in \mathcal{Q}(A \restriction_{X'})$ then also $f \in \mathcal{Q}(A_{obs}(\Sigma) \restriction_{X'})$.

Consider an arbitrary $f \in \mathcal{Q}(A \restriction_{X'})$. Then $f = \frac{\Psi([f_1])}{\Psi([f_2])}$ where $f_1, f_2 \in A$, $f_2 \neq 0$. Due to the observability of the system Σ, $\frac{f_1}{f_2} \in Q = Q_{obs}(\Sigma) = \mathcal{Q}(A_{obs}(\Sigma))$ and therefore there exist $g_1, g_2 \in A_{obs}(\Sigma)$, $g_2 \neq 0$ such that

$$f_2 g_1 = f_1 g_2. \tag{13}$$

This implies that $g_1 \upharpoonright_{X'}$ and $g_2 \upharpoonright_{X'}$ are both elements of $A_{obs}(\Sigma) \upharpoonright_{X'}$, $g_2 \upharpoonright_{X'} \neq [0]$ and consequently that $g = \frac{\Psi([g_1])}{\Psi([g_2])} = \frac{g_1 \upharpoonright_{X'}}{g_2 \upharpoonright_{X'}} \in \mathcal{Q}(A_{obs}(\Sigma) \upharpoonright_{X'})$. Because the bijection Ψ preserves sums and products we get from (13) that $\Psi([f_2])\Psi([g_1]) = \Psi([f_2 g_1]) = \Psi([f_1 g_2]) = \Psi([f_1])\Psi([g_2])$ which implies that $f = \frac{\Psi([f_1])}{\Psi([f_2])} = \frac{\Psi([g_1])}{\Psi([g_2])} = g$. Hence, $f \in \mathcal{Q}(A_{obs}(\Sigma) \upharpoonright_{X'})$ for arbitrary $f \in \mathcal{Q}(A \upharpoonright_{X'})$ and therefore the rational system Σ' is observable.

$(b) \Rightarrow (c)$ We assume that p has a canonical rational realization $\Sigma = (X, f, h, x_0)$. The existence of rational realization of p implies, due to Theorem 4.9, that $Q_{obs}(p)$ is finitely generated.

$(c) \Rightarrow (a)$ We assume that the observation field $Q_{obs}(p)$ of p is finitely generated. By Proposition 4.7, $Q_{obs}(p)$ is finitely generated by the elements from $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$. By the definition of observation field, $Q_{obs}(p)$ is closed with respect to D_α derivations $\alpha = (\alpha_1, \ldots, \alpha_i), i \in \mathbb{N}, \alpha_j \in U, j = 1, \ldots, i$. So, let $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$ where $\varphi_i : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}, i = 1, \ldots, k$ are analytic functions. To prove that there exists an observable rational realization of p, we construct a rational realization $\Sigma = (X, f, h, x_0)$ of p such that $Q_{obs}(\Sigma) = Q$ where Q denotes the field of rational functions on X.

The field F defined as $F = Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$ fulfills the conditions (i) - (iii) of Proposition 4.8. By following the proof of Proposition 4.8, we construct a rational realization $\Sigma = (X, f, h, x_0)$ of p as

$$\begin{aligned} X &= \mathbb{R}^k, \\ f_\alpha &= \sum_{i=1}^{k} v_i^\alpha \frac{\partial}{\partial x_i}, \\ h_j(x_1, \ldots, x_k) &= w_j(x_1, \ldots, x_k), \quad j = 1 \ldots r, \\ x_0 &= (\varphi_1(e), \ldots, \varphi_k(e)). \end{aligned}$$

This realization is such that

$$\begin{aligned} p_j &= w_j(\varphi_1, \ldots, \varphi_k), j = 1 \ldots r, \text{ and} \\ D_\alpha \varphi_i &= v_i^\alpha(\varphi_1, \ldots, \varphi_k), i = 1 \ldots k. \end{aligned}$$

We prove that the constructed realization is observable.

Because $X = \mathbb{R}^k$, we get for the field Q of rational functions on X that $Q = \mathbb{R}(x_1, \ldots, x_k)$. To consider h_j and $f_\alpha h_j$, $j = 1 \ldots r$ is the same as to consider p_j and $D_\alpha p_j$ but just in different coordinates. Therefore, $Q_{obs}(\Sigma)$, as a field of quotients of the smallest subalgebra of Q containing all $h_j, j = 1 \ldots r$ and its D_α derivations, equals $\mathbb{R}(x_1, \ldots, x_k)$ in analogy to the relation $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$. That means that

$$Q_{obs}(\Sigma) = \mathbb{R}(x_1, \ldots, x_k) = Q,$$

proving the observability of a rational realization Σ of p. $\square$

Corollary 5.2 *Let p be an analytic response map. According to Theorem 4.9 and Theorem 5.1 the following statements are equivalent:*

(i) p is realizable by a rational system,

(ii) p has rational realization which is observable,

(iii) p has rational realization which is canonical, thus both observable and controllable.

6 Minimal rational realizations

Minimal realizations within the class of linear systems realizing a given response map are defined as linear realizations of this map for which the state space dimension is minimal overall such linear systems. For linear systems, the propery of being minimal in this sense can be proven to be equivalent to the propery of being observable and controllable.

Many people tried to extend the same concept of minimality to the world of general nonlinear systems (see for example [28, 29, 42, 43]).

Minimality of the polynomial realizations was firstly defined in [40] by E.D. Sontag in the discrete-time case as minimal-dimensional realization, i.e. as a realization having the state space of minimal dimension within all realizations, where the dimension of the state space X was understood as the transcendence degree of the polynomial functions on X. Afterwards Z. Bartosiewicz in [6], generalizing discrete-time polynomial case to continuous-time case, stays with the concept of transcendence degree introduced by Sontag for minimal dimensionality to define so-called algebraically minimal polynomial realizations. He proves that these systems are algebraically observable and controllable.

To define minimality for rational realizations, we were motivated by the papers [5, 6]. Analogous to realization theory of linear systems, we want the property of being a minimal rational realization to be equivalent to being a canonical (observable and controllable) rational realization and a minimal-dimensional rational realization. These requirements with an additional assumptions are fulfilled by defining a minimal realization as a realization whose state space dimension equals the transcendence degree of the observation field of the map realized by the system. We prove that minimal realizations are unique up to a birational equivalence and that canonical realizations are birationally equivalent.

We are aware of two papers, [5] and [47] concerning rational systems. In [47] the problem of minimality is not considered. In [5], the problem of rational realization is not considered and thus neither the problem of minimal rational realization. In spite of this, one can observe analogies between the minimal dimension of rational system to which a $\mathcal{C}^\infty$ system can be immersed, which is studied in [5], and the definition of minimal rational systems we propose.

The short history of the development of knowledge of minimality is sketched in [41].

Definition 6.1 *Let $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$ be an analytic response map. We call a rational realization $\Sigma = (X, f, h, x_0)$ of p a* minimal realization *of p if $\dim X =$ trdeg $Q_{obs}(p)$.*

We remark that due to the definition, all minimal rational realizations of the same response map have the same dimension.

Lemma 6.2 *Let Σ be a rational realization of an analytic response map p. Then*

$$\text{trdeg } Q_{obs}(p) \leq \text{trdeg } Q_{obs}(\Sigma).$$

Proof: From Proposition 4.6 (i) and from Proposition 2.14, it follows that trdeg $Q_{obs}(p) =$ trdeg $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$. Moreover

$$\text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \leq \text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)) = \text{trdeg } Q_{obs}(\Sigma).$$

Therefore trdeg $Q_{obs}(p) \leq$ trdeg $Q_{obs}(\Sigma)$ which was to be proved. □

Hence, all rational realizations of a response map have the dimension higher (not necessarilly strictly higher) than the dimension of a minimal rational realization of the same map.

Minimal versus canonical realizations

In this subsection we relate the properties of rational realizations of being minimal, observable, and controllable.

Firstly, we prove that canonicity of a rational realization implies its minimality and that even a rational realization which is controllable but not observable can be minimal.

Proposition 6.3 *Let $\Sigma = (X, f, h, x_0)$ be a canonical (both observable and controllable) rational realization of an analytic response map p. Then Σ is also minimal rational realization of p.*

Proof: We assume that Σ is a canonical rational realization of an analytic response map p. Hence, Σ is controllable, which means, due to Definition 3.15, that $X = Z - cl\mathcal{R}(x_0)$. Then, from the properties of the map $\tau^*_{ext} : A_{obs}(\Sigma) \to \mathcal{A}_{obs}(p)$ which is defined in Proposition 4.4 as $\tau^*_{ext}(\varphi) = \varphi \circ \tau$ for all $\varphi \in A_{obs}(\Sigma)$ we know that

$$\text{Ker } \tau^*_{ext} = \{f \in A_{obs}(\Sigma) | f = 0 \text{ on } \mathcal{R}(x_0)\} = \{f \in A_{obs}(\Sigma) | f = 0 \text{ on } X\}.$$

Therefore $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext} = A_{obs}(\Sigma)$ and consequently $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) = Q_{obs}(\Sigma)$. Thus

$$\text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) = \text{trdeg } Q_{obs}(\Sigma). \tag{14}$$

Because we know that the map $\widehat{\tau^*} : \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \to Q_{obs}(p)$ defined in Proposition 4.6 (i) is an isomorphism, it follows from Proposition 2.14 that

$$\text{trdeg } Q_{obs}(p) = \text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}). \tag{15}$$

Because Σ is also an observable rational realization of p, by Definition 3.17, $Q_{obs}(\Sigma) = Q$ and trdeg $Q_{obs}(\Sigma) =$ trdeg Q. Then from the relations (14) and (15) it follows that

$$\text{trdeg } Q_{obs}(p) = \text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) = \text{trdeg } Q_{obs}(\Sigma) = \text{trdeg } Q.$$

Hence trdeg $Q_{obs}(p) =$ trdeg Q and finally, by the definition of the dimension of a variety, we get

$$\dim X = \text{trdeg } Q = \text{trdeg } Q_{obs}(p)$$

which proves that the system Σ is a minimal rational realization of p. □

From the proposition above we can conclude that the existence of a rational realization of a response map implies the existence of minimal rational realization for this map.

Theorem 6.4 *If an analytic response map p has a rational realization, then p has also a minimal rational realization.*

Proof: This is a direct consequence of Corollary 5.2 and Proposition 6.3. □

Proposition 6.5 *Let Σ be a rational realization of an analytic response map p which is controllable but which is not observable. If the elements of $Q \setminus Q_{obs}(\Sigma)$ are algebraic over $Q_{obs}(\Sigma)$, then Σ is minimal.*

Proof: Because Σ is not observable, we get that $Q_{obs}(\Sigma) \subsetneq Q$. In spite of this, since the elements of $Q \setminus Q_{obs}(\Sigma)$ are algebraic over $Q_{obs}(\Sigma)$,

$$\text{trdeg } Q_{obs}(\Sigma) = \text{trdeg } Q. \tag{16}$$

From the controllability we get in the same way as in the proof of Proposition 6.3 (by applying Definition 3.12, Definition 3.15, Proposition 4.4, Proposition 4.6) that

$$\text{trdeg } Q_{obs}(\Sigma) = \text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) = \text{trdeg } Q_{obs}(p). \tag{17}$$

Therefore by the definition of the dimension of a variety and by (16) and (17) we get that trdeg $Q_{obs}(p) =$ trdeg $Q = \dim X$ and thus that the rational realization Σ of p is minimal. □

For the reversed implication, i.e. observability and controllability being determined by minimality, we study two problems. One is whether the minimality of a rational realization means that the realization is already observable, and the other one is whether a minimal rational realization is controllable.

The next proposition answers the first question, whether a minimal rational realization is observable.

Proposition 6.6 *Let $\Sigma = (X, f, h, x_0)$ be a minimal rational realization of an analytic response map p and let Q denote the field of rational functions on X. If the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$, then the rational realization Σ is observable.*

Proof: Let $\Sigma = (X, f, h, x_0)$ be a minimal rational realization of an analytic response map p such that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$.

From the minimality of Σ it follows that trdeg Q = trdeg $Q_{obs}(p)$. From the part (i) of Proposition 4.6 we know that there exists an isomorphism $\widehat{\tau^*}$: $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \to Q_{obs}(p)$. Therefore, by Proposition 2.14

$$\text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) = \text{trdeg } Q. \tag{18}$$

Because $Q_{obs}(\Sigma)$ is a subfield of Q, it follows from Proposition 2.12 that

$$\text{trdeg } Q_{obs}(\Sigma) \leq \text{trdeg } Q. \tag{19}$$

Consequently, by (18) and (19),we obtain

$$\text{trdeg } Q = \text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \leq \text{trdeg } Q_{obs}(\Sigma) \leq \text{trdeg } Q. \tag{20}$$

Hence, trdeg Q = trdeg $Q_{obs}(\Sigma)$. By the assumption that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$ and by Proposition 2.13, we get that $Q = Q_{obs}(\Sigma)$ which proves the observability of Σ. □

From Proposition 6.6 it is obviously true that if a rational realization Σ of a response map p is not observable then Σ is not minimal or the elements of $Q \setminus Q_{obs}(\Sigma)$ are algebraic over $Q_{obs}(\Sigma)$. In the following proposition we prove that if a rational realization Σ is not observable and if the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$, then Σ is not minimal.

Therefore, the proposition below says that if we know that a rational realization Σ is minimal and that the elements of $Q \setminus Q_{obs}(\Sigma)$ are algebraic over $Q_{obs}(\Sigma)$ then we are still not able to say, without additional information, whether Σ is observable or not.

Proposition 6.7 *Let Σ be a rational realization of an analytic response map p which is not observable. If the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$, then Σ is not minimal.*

Proof: To prove that the rational realization Σ of p is not minimal we need to prove that $\dim X$ = trdeg $Q \neq$ trdeg $Q_{obs}(p)$. Thus, according to Lemma 6.2, we need to prove that trdeg $Q_{obs}(p) <$ trdeg Q.

Since the rational system Σ is not observable, $Q_{obs}(\Sigma) \subsetneqq Q$. Therefore, because of the assumption that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$, we get that

$$\text{trdeg } Q_{obs}(\Sigma) < \text{trdeg } Q. \tag{21}$$

(Note that trdeg $F \leq$ trdeg G for arbitrary subfield F of a field G.)

Because trdeg $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \leq$ trdeg $Q_{obs}(\Sigma)$ and because by Proposition 4.6 (i) there is an isomorphism $\widehat{\tau^*} : \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \to Q_{obs}(p)$, we get by Proposition 2.14 that

$$\text{trdeg } Q_{obs}(p) = \text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \leq \text{trdeg } Q_{obs}(\Sigma). \tag{22}$$

From (21) and (22) it follows that trdeg $Q_{obs}(p) <$ trdeg $Q = \dim X$ which proves that Σ is not minimal. $\square$

In the rest of this subsection we assume that the state space X of a rational realization is the trivial variety $\mathbb{R}^n$ for $n \in \mathbb{N}$. Then the field Q of rational functions on $\mathbb{R}^n$ is generated by n generators and moreover trdeg $Q = n$. The results below, concerning the question whether minimality implies controllability, are valid for all rational realizations for which the fields of rational functions on their state spaces are such that the number of generators of that fields equals their transcendence degree. The extension to an arbitrary irreducible real affine variety is still under investigation.

Lemma 6.8 *Let F be a finitely generated field such that $F = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$ and* trdeg $F = k$. *Consider a field homomorphism ι on F. We denote by G the field $\iota(F) = \mathbb{R}(\iota\varphi_1, \ldots, \iota\varphi_k)$, i.e. $G = \iota(F)$. If there exists a non-zero $\varphi \in F$ such that $\iota\varphi = 0$, then* trdeg $G <$ trdeg F.

Proof: Let $\varphi \in F$ be such that $\iota\varphi = 0$ and $\varphi = \frac{p(\varphi_1,\ldots,\varphi_k)}{q(\varphi_1,\ldots,\varphi_k)}$ where p, q are non-zero polynomials with real coefficients. Then

$$0 = \iota(\varphi) = \iota\frac{p(\varphi_1, \ldots, \varphi_k)}{q(\varphi_1, \ldots, \varphi_k)} = \frac{p(\iota\varphi_1, \ldots, \iota\varphi_k)}{q(\iota\varphi_1, \ldots, \iota\varphi_k)}, \text{ and consequently}$$
$$0 = p(\iota\varphi_1, \ldots, \iota\varphi_k).$$

Since p is a non-zero polynomial in k variables with real coefficients, it implies that the generators $\iota\varphi_1, \ldots, \iota\varphi_k$ of the field G are algebraically dependent. Because of this dependence and because, by [35, Chapter X, Theorem 1], a transcendence basis can be chosen as a subset of generators,

$$\text{trdeg } G \leq k - 1 < k = \text{trdeg } F.$$

Thus trdeg $G <$ trdeg F which completes the proof. $\square$

Proposition 6.9 *Let $\Sigma = (X, f, h, x_0)$ be a minimal rational realization of an analytic response map p such that $X = \mathbb{R}^n$ for $n \in \mathbb{N}$. Let Q denote the field of rational functions on X. If the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$, then the rational realization Σ is controllable.*

Proof: Consider a minimal rational realization $\Sigma = (X, f, h, x_0)$ of an analytic response map p such that $X = \mathbb{R}^n$ for $n \in \mathbb{N}$. Let Q denote the field of rational functions on X. Because $X = \mathbb{R}^n$, there exist the functions $\varphi_1, \ldots, \varphi_n \in Q$ such that $Q = \mathbb{R}(\varphi_1, \ldots, \varphi_n)$, and moreover we know that trdeg $Q = n$. We assume that the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$.

We denote by A the algebra of polynomials on X and by Q the field of rational functions on X. The rational realization Σ is controllable if the reachable set $\mathcal{R}(x_0)$ is Z-dense in X. Note that generally $Z - cl\mathcal{R}(x_0) \subseteq X$. Then, to prove that $X = Z - cl\mathcal{R}(x_0)$, we need to show that $Z - cl\mathcal{R}(x_0) \supseteq X$.

Because Σ is a minimal rational realization, trdeg $Q =$ trdeg $Q_{obs}(p)$. This implies, by Proposition 4.6 (i) and by Proposition 2.14, that

$$\text{trdeg } Q = \text{trdeg } \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^*). \tag{23}$$

Because Σ is minimal and because the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$, we get from Proposition 6.6 that Σ is observable. Therefore

$$Q = Q_{obs}(\Sigma).$$

Consider the map $\iota : Q \to \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^*)$ defined as $\iota(\varphi) = \iota(\frac{\varphi_{num}}{\varphi_{den}}) = \frac{[\varphi_{num}]}{[\varphi_{den}]}$ for $\varphi \in Q = Q_{obs}(\Sigma)$. The map ι is a homomorphism of the fields Q and $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau_{ext}^*)$. After applying Lemma 6.8 to the field Q and to the homomorphism ι, we get by considering (23) that for every non-zero element $\varphi \in Q = Q_{obs}(\Sigma)$ it holds that $\iota\varphi \neq [0]$. This means that if the element $\varphi \in Q$ is not identically zero on X, then $\iota(\varphi) = \frac{[\varphi_{num}]}{[\varphi_{den}]}$ is such that $[\varphi_{num}] \neq [0] = \text{Ker } \tau_{ext}^*$ implying that φ_{num} and hence also φ is not identically zero on $\mathcal{R}(x_0)$. Since $A \subseteq Q$, the same is true for the polynomials $\varphi \in A$. It can be written in the following form:

$$\forall \varphi \in A : \varphi = 0 \text{ on } \mathcal{R}(x_0) \Rightarrow \varphi = 0 \text{ on } X. \tag{24}$$

We denote by I the ideal of polynomials determining the variety X, i.e. I is such that $X = \{x \in \mathbb{R}^n | f(x) = 0 \text{ for all } f \in I\}$, and we consider the map $\tau^* : A \to \mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ defined in Definition 3.12. Then, from (24) it follows that

$$\text{Ker } \tau^* = \{f \in A | f = 0 \text{ on } \mathcal{R}(x_0)\} \subseteq \{f \in A | f = 0 \text{ on } X\} = I$$

and thus that

$$\{x \in X | f(x) = 0 \text{ for all } f \in \text{Ker } \tau^*\} \supseteq \{x \in X | f(x) = 0 \text{ for all } f \in I\}$$

which means that $Z - cl\mathcal{R}(x_0) \supseteq X$. This concludes the proof. $\square$

Remark 6.10 *According to the proof of Proposition 6.9, if* $\Sigma = (X, f, h, x_0)$ *is a minimal rational realization of an analytic response map* p *such that* $X = \mathbb{R}^n$ *for* $n \in \mathbb{N}$ *and if* Σ *is observable, then* Σ *is controllable.*

Theorem 6.11 *Let* $\Sigma = (X, f, h, x_0)$ *be a rational realization of an analytic response map* p *such that* $X = \mathbb{R}^n$ *for* $n \in \mathbb{N}$ *and such that the elements of* $Q \setminus Q_{obs}(\Sigma)$ *are not algebraic over* $Q_{obs}(\Sigma)$*. Then* Σ *is canonical if and only if* Σ *is minimal.*

Proof: This follows directly from the propositions 6.6, 6.9, and 6.3. □

Minimal versus minimal-dimensional realizations

Definition 6.12 *A rational realization of an analytic response map is called* minimal-dimensional *if its state-space dimension is minimal, i.e. there does not exist another rational realization of the same map such that its state space has a strictly lower dimension.*

Theorem 6.13 *A rational realization* Σ *of an analytic response map* p *is minimal if and only if* Σ *is minimal-dimensional.*

Proof: ($\Leftarrow$) Let Σ be a minimal-dimensional rational realization of p. If $\dim X = \text{trdeg } Q_{obs}(p)$, then Σ is also minimal and we are done. Hence, assume that $\dim X \neq \text{trdeg } Q_{obs}(p)$. According to Lemma 6.2, we can equivalently assume that $\dim X > \text{trdeg } Q_{obs}(p)$. From Theorem 6.4 we know that there exists a minimal rational realization Σ', i.e. a rational realization Σ' such that $\dim X' = \text{trdeg } Q_{obs}(p)$. But then $\dim X > \dim X'$ which contradicts the minimal-dimensionality of Σ. Therefore Σ has to be such that $\dim X = \text{trdeg } Q_{obs}(p)$.

($\Rightarrow$) It follows directly from Lemma 6.2. □

7 Birational equivalence of rational realizations

In this section we study the relation between minimality and birational equivalence of rational realizations. We prove that every rational realization of a response map which is birationally equivalent to a minimal rational realization of the same map, is itself minimal. On the other hand, we show that canonical rational realizations are birationally equivalent. Therefore we know that minimal rational realizations are birationally equivalent if they are canonical. This is for example in the case when the assumptions of Theorem 6.11 are fulfilled.

Note that birational equivalence of irreducible varieties is a weaker equivalence relation than isomorphism. That means that the set of varieties birationally equivalent to a given variety contains many different nonisomorphic varieties.

The terminology of rational mappings between varieties and birational equivalence of varieties is adopted from [13]. For the completeness of the paper we recall some definitions and statements below.

Definition 7.1 *[13, Chapter 5.5, Definition 4] Let $X \subseteq \mathbb{R}^m$ and $X' \subseteq \mathbb{R}^n$ be irreducible real affine varieties. A* rational mapping *from X to X' is a function ϕ represented by*

$$\phi(x_1, \ldots, x_m) = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)),$$

where $f_i \in \mathbb{R}(x_1, \ldots, x_m)$ are such that ϕ is defined at some point of X, and if ϕ is defined at the point $(a_1, \ldots, a_m) \in X$, then $\phi(a_1, \ldots, a_m) \in X'$.

Similarly as the rational functions on a variety, rational mappings between varieties do not have to be defined everywhere.

Definition 7.2 *[13, Chapter 5.5, Definition 9(i)] Two irreducible varieties X and X' are* birationally equivalent *if there exist rational mappings $\phi : X \to X'$, $\psi : X' \to X$ such that $\phi \circ \psi = 1_Y$ and $\psi \circ \phi = 1_X$.*

Theorem 7.3 *([13, Chapter 9.5, Corollary 7]) Let the irreducible real affine varieties X and X' be birationally equivalent. Then* $\dim X = \dim X'$.

Theorem 7.4 *([13, Chapter 5.5, Theorem 10]) Let X and X' be irreducible real affine varieties and let Q and Q' denote the field of rational functions on X and X', respectively. Then the varieties X and X' are birationally equivalent if and only if there exists an isomorphism of the fields Q and Q' which is the identity on $\mathbb{R}$.*

The next definition defining birationally equivalent rational realizations can be found in [5] as Definition 8.

Definition 7.5 *Let $\Sigma = (X, f, h, x_0)$ and $\Sigma' = (X', f', h', x'_0)$ be rational realizations of the same analytic response map p. We say that Σ and Σ' are* birationally equivalent *if*

(i) the state spaces X and X' are birationally equivalent, which means that there exist rational maps $\phi : X \to X'$, $\psi : X' \to X$ such that $\phi \circ \psi = 1_{X'}$ and $\psi \circ \phi = 1_X$,

(ii) $h'\phi = h$, and

(iii) $f_\alpha(\varphi \circ \phi) = (f'_\alpha \varphi) \circ \phi$ for $\varphi \in Q'$, $\alpha \in U$.

Theorem 7.6 *Let Σ be a minimal rational realization of an analytic response map p. Then every rational realization of the same map p which is birationally equivalent to Σ is minimal.*

Proof: Let Σ be a minimal rational realization of an analytic response map p and let Σ' be a rational realization of the same map p which is birationally equivalent to Σ.

From the birational equivalence of Σ and Σ' and from Theorem 7.3, $\dim X = \dim X'$. Because Σ is minimal, $\dim X' = \dim X = \text{trdeg } Q_{obs}(p)$. Therefore the system Σ' is minimal as well. $\square$

Theorem 7.7 *Let $\Sigma = (X, f, h, x_0)$ and $\Sigma' = (X', f', h', x'_0)$ be canonical rational realizations of the same analytic response map p. Then Σ and Σ' are birationally equivalent.*

Proof: Let $\Sigma = (X, f, h, x_0)$ and $\Sigma' = (X', f', h', x'_0)$ be canonical rational realizations of the same response map p.

From Proposition 4.6 it follows that the maps $\widehat{\tau^*} : \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \to Q_{obs}(p)$ and $\widehat{\tau'^*} : \mathcal{Q}(A_{obs}(\Sigma')/\text{Ker } \tau'^*_{ext}) \to Q_{obs}(p)$ are isomorphisms.

We define the map $\Psi : A_{obs}(\Sigma) \to A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ as $\Psi(f) = [f]$ such that $[f] = [g]$ if $f - g \in \text{Ker } \tau^*_{ext}$. The map Ψ is a surjective homomorphism. We prove that it is even injective.

Consider $f, g \in A_{obs}(\Sigma)$ such that $\Psi(f) = \Psi(g)$. Thus $[f] = [g]$ which means that $f - g \in \text{Ker } \tau^*_{ext} = \{f \in A_{obs}(\Sigma) | f = 0 \text{ on all } \mathcal{R}(x_0)\}$ and therefore $f - g = 0$ on $\mathcal{R}(x_0)$. Because the realization Σ is controllable, we know that $Z - cl\mathcal{R}(x_0) = X$. Hence the controllability implies that a polynomial φ is zero on all $\mathcal{R}(x_0)$ if and only if $\varphi \in I$ where I is the ideal of polynomials determining the variety X. Therefore $f - g = 0$ on $\mathcal{R}(x_0)$ means that $f - g \in I$ which means that $f - g = 0$ on X. Finally, $f = g$ and the surjective homomorphism Ψ is also injective, i.e. Ψ is an isomorphism.

Because $A_{obs}(\Sigma)$ and $A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ are integral domains and because $\Psi : A_{obs}(\Sigma) \to A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}$ is an isomorphism, we can extend Ψ to the isomorphism of the fields $Q_{obs}(\Sigma)$ and $\mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$. Hence

$$\Psi : Q_{obs}(\Sigma) \to \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext})$$

is an isomorphism.

We define the map $\Psi' : A_{obs}(\Sigma') \to A_{obs}(\Sigma')/\text{Ker } \tau'^*_{ext}$ analogically as we have defined the map Ψ. In the same way we also extend the map Ψ' to the isomorphism of the fields $Q_{obs}(\Sigma')$ and $\mathcal{Q}(A_{obs}(\Sigma')/\text{Ker } \tau'^*_{ext})$. Hence

$$\Psi' : Q_{obs}(\Sigma') \to \mathcal{Q}(A_{obs}(\Sigma')/\text{Ker } \tau'^*_{ext})$$

is an isomorphism.

From the observability of the realizations Σ and Σ' we get that $Q_{obs}(\Sigma) = Q$ and $Q_{obs}(\Sigma') = Q'$.

Because the maps $\Psi, \Psi', \widehat{\tau^*}, \widehat{\tau'^*}$ are isomorphisms, the map $i : Q' \to Q$ defined as $i = \Psi^{-1} \circ (\widehat{\tau^*})^{-1} \circ \widehat{\tau'^*} \circ \Psi'$ is also an isomorphism. The inverse of i is $i^{-1} = \Psi'^{-1} \circ (\widehat{\tau'^*})^{-1} \circ \widehat{\tau^*} \circ \Psi : Q \to Q'$. See the figure below for a diagram desribing the relations between $i, \Psi, \Psi', \widehat{\tau^*}, \widehat{\tau'^*}$.

$$
\begin{tikzcd}
Q = Q_{obs}(\Sigma) \arrow[r, "\Psi"] \arrow[dd, shift left, "i^{-1}"] & \mathcal{Q}(A_{obs}(\Sigma)/\text{Ker } \tau^*_{ext}) \arrow[dr, "\widehat{\tau^*}"] & \\
 & & Q_{obs}(p) \\
Q' = Q_{obs}(\Sigma') \arrow[r, "\Psi'"'] \arrow[uu, shift left, "i"] & \mathcal{Q}(A_{obs}(\Sigma')/\text{Ker } \tau'^*_{ext}) \arrow[ur, "\widehat{\tau'^*}"'] &
\end{tikzcd}
$$

From the proof of Theorem 7.4 (see [13, Chapter 5.5, Theorem 10]) we get that there exists a birational mapping $\iota : X \to X'$ such that $\iota^* = i$ and ι is an isomorphism. By a birationality we mean that ι and ι^{-1} are both rational mappings. The map ι^* is defined as $\iota^*(\varphi) = \varphi \circ \iota$ for $\varphi \in Q'$. Note that $(\iota^{-1})^* = \iota^{-1}$.

So, the rational mappings $\iota : X \to X'$ and $\iota^{-1} : X' \to X$ are such that $\iota \circ \iota^{-1} = 1_{X'}$ and $\iota^{-1} \circ \iota = 1_X$. Therefore, the varieties X, X' are birationally equivalent.

To prove that not just the varieties X and X' but even the systems Σ and Σ' are birationally equivalent, we show that $h'\iota = h$ and $f_\alpha(\varphi \circ \iota) = (f'_\alpha \varphi) \circ \iota$ for $\varphi \in Q'$, $\alpha \in U$. We will check these two equalities be applying the properties of the isomorphisms $\Psi, \Psi', \widehat{\tau^*}, \widehat{\tau'^*}, \iota$ and ι^*.

Let us consider arbitrary $\varphi \in Q'$, $\alpha \in U$. Then

$$
h' \circ \iota \quad = \quad \iota^* h' = (\Psi^{-1} \circ (\widehat{\tau^*})^{-1} \circ \widehat{\tau'^*} \circ \Psi')h' = (\Psi^{-1} \circ (\widehat{\tau^*})^{-1})(p) = h,
$$

and

$$
\begin{aligned}
f_\alpha(\varphi \circ \iota) &= f_\alpha(\iota^* \varphi) = f_\alpha((\Psi^{-1} \circ (\widehat{\tau^*})^{-1} \circ \widehat{\tau'^*} \circ \Psi')\varphi) \\
&= \Psi^{-1} \circ (\widehat{\tau^*})^{-1}(D_\alpha(\widehat{\tau'^*}\Psi'\varphi)) = (\Psi^{-1} \circ (\widehat{\tau^*})^{-1} \circ \widehat{\tau'^*} \circ \Psi')(f'_\alpha \varphi) \\
&= \iota^*(f'_\alpha \varphi) = (f'_\alpha \varphi) \circ \iota.
\end{aligned}
$$

Finally, the systems Σ and Σ' are birationally equivalent. $\square$

8 Computational algebra for realization

The application of rational realization theory for systems biology and for engineering requires procedures to check controllability and observability of rational systems. For example, in metabolic networks one is provided a rational system from first principles and wants to know whether such a system is controllable or observable. This is not obvious because of the modeling assumptions and because of the modular way these networks are formulated.

In this section we discuss how the rational realization theory developed before could be used for developing the algorithms/procedures for checking controllability, observability and minimality of rational systems. We shortly mention the possibility of constructing rational realizations. However, there are not yet ready-made algorithms available.

In theoretical computer science one speaks of an algorithm if it can be proven that the corresponding procedure, if it runs on a Turing machine, terminates in a finite number of steps. Since the termination in a finite number of steps for the procedures of this section is not yet clear, we prefer to speak of procedures rather than of algorithms.

Differential and computational algebra

A rational system is specified in terms of a set of differential equations and of the read-out equation. Then the main algebraic objects of concern for the procedures for checking observability and minimality of rational systems and realizations are algebras and fields of polynomial, rational or analytic functions (analytic in the sense of Definition 3.9) over the real numbers. Since these algebraic objects are assumed to be closed with respect to certain derivations, we could speak about differential algebras and differential fields. Therefore the differential algebra developed by J.F. Ritt [39], and by E.R. Kolchin [33] is useful. An introduction to the differential algebra is provided by [31]. See also the recent work of F. Boulier [10].

For problems of control and system theory several researchers developed a framework using differential algebra. An early reference is that of M. Fliess, see [19], and the references therein. S. Diop and M. Fliess developed characteristic sets for nonlinear systems, see [16, 17]. Identifiability of systems was developed by F. Ollivier, see [38]. Equivalences of rational systems for identifiability were treated in [2]. T. Glad and K. Forsman developed differential algebra for control and system theory, see [21, 22]. An expository paper about these developments is the paper [20] by M. Fliess and T. Glad.

Another useful field for developing the procedures or even algorithms for checking observability and controllability of rational systems, and minimality of rational realizations is computational algebra. The major tool is the Gröbner basis method. For the references on computational algebra see for example the handbook [23] and the text books [8, 13, 14, 24].

In control and system theory the majority of the applications concern the field of the real numbers. Therefore the restriction of our attention to the field of real numbers is not restrictive from this point of view. Indeed, because the field of real numbers is not algebraically closed, considering the real numbers instead of, for example, complex numbers just makes the theory more difficult. The algorithmic approach to real algebraic geometry is treated in [7]. For a more theoretical reference see [9].

Procedures

Based on the preceding sections we propose the procedures for checking observability of rational systems and minimality of rational realizations. We discuss the possibility of developing a procedure for checking controllability of rational systems and a procedure to construct a rational realization of a given analytic response map.

Observability The following procedure for checking observability of a rational systems is based just on the definition of observability of rational systems, see Definition 3.17.

Procedure 8.1 Checking observability of a rational system. *Consider a rational system $\Sigma = (X, f, h)$.*

1. *Calculate the observation algebra $A_{obs}(\Sigma)$ of a rational system Σ.*
2. *Calculate the observation field $Q_{obs}(\Sigma)$ as a field of fractions of $A_{obs}(\Sigma)$.*
3. *Compute a set B of generators of algebra of polynomials on X, i.e. a set of generators of a quotient ring $\mathbb{R}[x_1, \ldots, x_n]/I$ where I is the ideal of polynomials vanishing on X.*
4. *Check whether $B \subseteq Q_{obs}(\Sigma)$.*
5. *If $B \subseteq Q_{obs}(\Sigma)$, then $Q_{obs}(\Sigma) = Q$ where Q is the field of rational functions on X. Thus, if $B \subseteq Q_{obs}(\Sigma)$ then the rational system is observable. Otherwise the system is not observable.*

We recall that to compute the observation algebra $A_{obs}(\Sigma)$ we need to construct the smallest algebra containing $h_i, i = 1, \ldots, r$ which is closed with respect to differentiation by the family of vector fields f. Hence we are computing a differential algebra. The fact that the observation field $Q_{obs}(\Sigma)$ is finitely generated and that a set B of generators of algebra of polynomials on X is a finite set could simplify the computations for the second and the third step of the procedure. The set B of polynomials on X could be computed by a Gröbner basis algorithm. The fourth step of the procedure above could be executed elementwise. The algorithms for checking whether an element of B (and therefore an element of the field Q of rational functions on X) is also an element of the field $Q_{obs}(\Sigma)$ are described in [36] and [37].

Minimality We could check minimality of a rational realization of a given analytic response map just by following the definitions.

Procedure 8.2 Checking minimality of a rational realization. *Consider a rational realization $\Sigma = (X, f, h, x_0)$ of a given analytic response map p.*

1. *Calculate the dimension* $\dim X$ *of an irreducible real affine variety X as the degree of the affine Hilbert polynomial of the corresponding ideal (ideal generated by the polynomials defining the variety X).*
2. *Compute the observation field $Q_{obs}(p)$ of a map p.*
3. *Calculate the transcendence degree of a field $Q_{obs}(p)$.*

4. *If* $\dim X = \operatorname{trdeg} Q_{obs}(p)$, *then the rational realization* Σ *of* p *is minimal. Otherwise* Σ *is not minimal.*

For this procedure we need to assume that the map p is available in analytic form. The first step of the procedure above is already implemented in Maple (see the command "HilbertDimension"). To calculate the observation field $Q_{obs}(p)$ of a response map p we proceed similarly as if we calculate the observation field of the system. Firstly we calculate the observation algebra of p as the smallest algebra containing $p_i, i = 1, \ldots, r$ which is closed with respect to certain derivations (see Definition 3.22) and then we build a field of fractions of its elements. The algorithms for computing the transcendence degree of field extensions of a field are presented in [36] and (of not necessarily purely transcendental field extensions) in [37]. There are other algorithms for the same problem which can be found in the references therein. These algorithms can be used for computing the transcendence degree of an observation field since an observation field as we defined it is a field extension of $\mathbb{R}$.

Controllability Again, the property of a rational system being controllable could be checked by definition (see Definition 3.15). Then we have to be able to construct the reachable sets of a rational system for all possible initial states, or in the case of a rational realization of a given response map, for an initial state. Afterwards we have to check whether these reachable sets are Z-dense in the state space.

Another way how to check controllability of a rational system is to look for an equivalent characterization of controllability. For rational realizations, Proposition 6.9 is such characterization. However, this proposition works just in the cases when $X = \mathbb{R}^n$ and when the elements of $Q \setminus Q_{obs}(\Sigma)$ are not algebraic over $Q_{obs}(\Sigma)$.

Construction of a rational realization For the procedure for the construction of a rational system realizing a given response map we follow the steps of the proof of Proposition 4.8.

Procedure 8.3 Constructing a rational realization of a given analytic response map. *Consider an analytic response map* $p : \widetilde{\mathcal{U}_{pc}} \to \mathbb{R}^r$.

1. *Check whether the observation field* $Q_{obs}(p)$ *is finitely generated (then* p *is realizable by a rational system, see Theorem 4.9).*

2. *Find the functions* $\varphi_1, \ldots, \varphi_k$ *from* $\mathcal{A}(\widetilde{\mathcal{U}_{pc}} \to \mathbb{R})$ *such that* $Q_{obs}(p) = \mathbb{R}(\varphi_1, \ldots, \varphi_k)$.

3. *Determine the functions* v_i^α *and* w_j *such that*

$$D_\alpha \varphi_i = v_i^\alpha(\varphi_1, \ldots, \varphi_k) \quad \textit{and} \quad p_j = w_j(\varphi_1, \ldots, \varphi_k)$$

for $i = 1, \ldots, k$, $j = 1, \ldots, r$, *and* $\alpha \in U$.

4. A rational realization of p is then

$$X = \mathbb{R}^k,$$
$$f_\alpha = \sum_{i=1}^{k} v_i^\alpha \frac{\partial}{\partial x_i}, \alpha \in U,$$
$$h_j(x_1, \dots, x_k) = w_j(x_1, \dots, x_k), j = 1 \dots r,$$
$$x_0 = (\varphi_1(e), \dots, \varphi_k(e)) \text{ where } e \text{ is an empty input, i.e. } T_e = 0.$$

Further research is needed for specifying more details of these procedures or developing new procedures. For that reason a deeper study of differential and computational algebra would be useful.

9 Concluding remarks

We have presented an algebraic approach to solve the realization problem for rational systems. We have characterized the response maps realizable by rational systems. We proved that once a response map is realized by a rational system, we can construct a rational realization which is observable or even canonical. Minimal rational realizations were defined as such rational realizations whose state spaces have dimension equal to the transcendence degree of the observation field of a considered response map. The relations between minimal, observable, and controllable rational realizations have been explored.

These results can help us to obtain computational tools for checking the properties, as observability and controllability of rational systems and minimality of rational realizations or even the tools for being able to automatically construct rational realizations of desired properties. The procedures to check observability of rational systems and minimality of rational realizations and the procedure to construct a rational realization for a given response map have been formulated but the paper does not provide the algorithms to check these properties.

Further research is required on the computational and differential algebra for the problem of realization of an analytic response map by a rational system. The realization theory for rational positive systems also requires attention. There are several issues to be overcome in that research.

Acknowledgements

The authors are very grateful to Mihály Petreczky for his careful reading of the previous versions of the paper and for his useful comments and advice.

References

[1] A. Agrachev and Y. Sachkov. *Control theory from the geometric viewpoint.* Springer, 2004.

[2] S. Audoly, G. Bellu, L. D'Agniò, M. P. Saccomani, and C. Cobelli. Global identifiability of nonlinear models of biological systems. *IEEE Trans. Biomedical Engineering*, 48:55–65, 2001.

[3] Z. Bartosiewicz. Realizations of polynomial systems. In M. Fliess and M. Hazenwinkel, editors, *Algebraic and Geometric Methods in Nonlinear Control Theory*, pages 45–54. D. Reidel Publishing Company, Dordrecht, 1986.

[4] Z. Bartosiewicz. Ordinary differential equations on real affine varieties. *Bulletin of the Polish Academy of Sciences Mathematics*, 35(1-2):13–18, 1987.

[5] Z. Bartosiewicz. Rational systems and observation fields. *Systems and Control Letters*, 9:379–386, 1987.

[6] Z. Bartosiewicz. Minimal polynomial realizations. *Mathematics of control, signals, and systems*, 1:227–237, 1988.

[7] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry.* Springer-Verlag, Berlin Heidelberg, 2003.

[8] T. Becker and V. Weispfenning. *Gröbner bases: A computational approach to commutative algebra.* Graduate Texts in Mathematics. Springer, Berlin, 1993.

[9] J. Bochnak, M. Coste, and M. F. Roy. *Real algebraic geometry.* Number 39 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin, 1998.

[10] F. Boulier. Récriture algébrique dans les systèmes d' équations différentielles polynomiales en vue d'applications dans les sciences du vivant. Mémoire d' habilitation à diriger des recherches, Universit des Sciences et Technologie de Lille - Lille I, Lille, 2006.

[11] N. Bourbaki. *Algebra II, Chapters 4-7.* Elements of mathematics. Springer-Verlag, 1990.

[12] P.M. Cohn. *Algebra, Volume 2.* John Wiley & Sons Ltd, 1977.

[13] D. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra.* Springer, 2nd edition, 1996.

[14] D.A. Cox, J.B. Little, and D.B. O'Shea. *Using algebraic geometry.* Number 185 in Graduate Texts in Mathematics. Springer, Berlin, 1998.

[15] P. d'Alessandro, A. Isidori, and A. Ruberti. Realization and structure theory of bilinear systems. *SIAM J. Control & Opt.*, 12:517–535, 1974.

[16] S. Diop. Elimination in control theory. *Math. Control, Signals Systems*, 4:17–32, 1991.

[17] S. Diop. Differential-algebraic decision methods and some applications to system theory. *Theoretical Computer Science*, 98:137–161, 1992.

[18] M. Feinberg. Chemical reaction network structure and the stability of complex isothermal reactors I. The deficiency zero and the deficiency one theorems. *Chemical Engineering Science*, 42:2229–2268, 1987.

[19] M. Fliess. Automatique et corps differentiels. *Forum Math.*, 1:227–238, 1989.

[20] M. Fliess and S.T. Glad. An algebraic approach to linear and nonlinear control. In H.L. Trentelman and J.C. Willems, editors, *Essays on control: Perspectives in the theory and its applications*, pages 223–267. Birkhäuser, Boston, 1993.

[21] K. Forsman. *Constructive commutative algebra in nonlinear control theory.* PhD thesis, Linköping University, Sweden, 1991.

[22] S.T. Glad. Differential algebraic modelling of nonlinear systems. In M.A. Kaashoek, A.C.M. Ran, and J.H. van Schuppen, editors, *Realization and modelling in system theory - Proc. International Symposium MTNS-89, Volume 1*, pages 97–105, Boston, 1990. Birkhäuser.

[23] J. Grabmeier, E. Kaltofen, and V. Weispfenning, editors. *Computer algebra handbook.* Springer, Berlin, 2003.

[24] A. Heck. *Introduction to Maple.* Springer, New York, 2nd. edition, 1996.

[25] F. Hynne, S. Dano, and P.G. Sorensen. Full-scale model of glycolysis in Saccharomyces cerevisiae. *Biophys. Chem.*, 94(1-2):121–163, 2001.

[26] N. Jacobson. *Basic Algebra II.* W. H. Freeman and company, San Francisco, 1980.

[27] N. Jacobson. *Basic Algebra I.* W. H. Freeman and company, New York, 2nd edition, 1985.

[28] B. Jakubczyk. Realization theory for nonlinear systems; three approaches. In M. Fliess and M. Hazenwinkel, editors, *Algebraic and Geometric Methods in Nonlinear Control Theory*, pages 3–31. D. Reidel Publishing Company, Dordrecht, 1986.

[29] B. Jakubczyk. Existence and uniqueness of realizations of nonlinear systems. *SIAM J. Control and Optimization*, 18(4):455–471, July 1980.

[30] R.E. Kalman. When is a linear control system optimal? *J. Basic Engineering*, 86:51–60, 1964.

[31] I. Kaplansky. *An introduction to differential algebra*. Publications de l'Institut de Mathematique de l'Universite de Nancano. Hermann, 1957.

[32] E. Klipp, R. Herwig, A. Kowald, C. Wierling, and H. Lehrach. *Systems biology in practice*. Wiley-VCH, Weinheim, 2005.

[33] E.R. Kolchin. *Differential algebra and algebraic groups.* Academic Press, New York, 1973.

[34] E. Kunz. *Introduction to commutative algebra and algebraic geometry.* Birkhäuser, Boston, 1985.

[35] S. Lang. *Algebra*. Addison-Wesley, Reading, Massachusetts, 1965.

[36] J. Müller-Quade and R. Steinwandt. Basic algorithms for rational function fields. *J. Symbolic Computation*, 27:143–170, 1999.

[37] J. Müller-Quade and R. Steinwandt. Gröbner bases applied to finitely generated field extensions. *J. Symbolic Computation*, 30:469–490, 2000.

[38] F. Ollivier. *Le problème de l'identifiabilité structurelle globale: approche théorique, méthodes effectives et bornes de complexité*. PhD thesis, École Polytechnique, 1990.

[39] J.F. Ritt. *Differential algebra.* Number 33 in American Mathematical Society Colloquium Publ. American Mathematical Society, 1950.

[40] E.D. Sontag. *Polynomial response maps.* Number 13 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, Heidelberg, 1979.

[41] E.D. Sontag. Spaces of observables in nonlinear control. In *Proceedings of the International Congress of Mathematicians*, volume 1,2, pages 1532–1545, 1995.

[42] H.J. Sussmann. Minimal realizations of nonlinear systems. In D. Mayne and R. Brockett, editors, *Geometric Methods in System Theory.* D. Reidel Publishing Company, 1973.

[43] H.J. Sussmann. Existence and uniqueness of minimal realizations of nonlinear systems. *Math. Systems Theory*, 10:263–284, 1977.

[44] J.H. van Schuppen. System theory of rational positive systems for cell reaction networks. In *Proceedings of MTNS, cd-rom*, 2004.

[45] Y. Wang. Analytic constraints and realizability for analytic input/output operators. *IMA Journal of Mathematical Control and Information*, 12(4):331–346, 1995.

[46] Y. Wang. Generalized input/output equations and nonlinear realizability. *International Journal of Control*, 64(4):615–629, 1996.

[47] Y. Wang and E.D. Sontag. Algebraic differential equations and rational control systems. *SIAM J. Control Optim.*, 30(5):1126–1149, 1992.

[48] Y. Wang and E.D. Sontag. Generating series and nonlinear systems: Analytic aspects, local realizability, and i/o representations. *Forum Math.*, 4(3):299–322, 1992.

[49] Y. Wang and E.D. Sontag. Orders of input/output differential equations and state-space dimensions. *SIAM J. Control Optim.*, 33(4):1102– 1126, 1995.

[50] D.J. Winter. *The structure of fields.* Springer-Verlag, New York, 1974.

[51] O. Zariski and P. Samuel. *Commutative algebra I, II.* Springer, 1958.